Budget Constraints in Combinatorial Clock Auctions[*]

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Abstract

In this chapter we analyze the Combinatorial Clock Auction (CCA) with budget-constrained bidders. By means of illustrative examples, we highlight positive and critical aspects of bidding under a budget in the clock and in the supplementary phase. Since the supplementary phase of a CCA without constraints from the clock phase is just a VCG auction, we also relate the CCA to the VCG. In the VCG auction, bidding under a budget constraint can be strategically complicated. However, on the positive side, the information revealed in the clock can facilitate bidding since bidders can compute upper (and lower) bounds on the final VCG price. This information might allow them to bid above budget without facing the risk of having to pay more than budget, which might lead to an efficient allocation. In one example we show that the clock might actually last longer under a budget constraint. More critical is that the information provided in the clock can actually be used to bid above budget in order to raise rivals' costs. There are asymmetric equilibria that have a Hawk-Dove flavor: the most aggressive bidder gets the highest surplus. Coordination issues can arise and in the case of mis-coordination bidders have to pay more than their budget.

1 Introduction

Combinatorial clock auctions (CCAs) are multi-object auctions where bidders make package bids in a clock phase followed by a supplementary round. CCAs have been recently used around the world to allocate spectrum frequencies for mobile telecommunication purposes. CCAs were introduced by Ausubel, Cramton and Milgrom (2006) and are the subject of quite a few recent investigations (see, e.g., Ausubel and Baranov (2014), Bichler et al. (2013), Janssen and Karamychev (2014), Knapek and Wambach (2012), Levin and Skrzypacz (2014), and papers in this volume).

One of the issues that is under-explored in the literature on CCAs is the impact of budget constraints on the bidding behavior of participating bidders and, consequently, on the efficiency properties of the auction. It is difficult to obtain direct evidence of the fact that

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budget constraints play an important role in real-life CCAs. It is also difficult to believe, however, that bidders in recent spectrum auctions have not been financially constrained. The amount of money typically paid is in the billions of Euros and even though a firm may think it will earn that money back in the years after the auction, it is likely it has to borrow the money in one way or the other. Also, casual empiricism suggests that share prices of companies participating in a long-lasting auction decline during and after the auction (*cf.* share prices of KPN in the Netherlands in 2012 and A1 in Austria in 2013).

It can be useful to distinguish between hard and soft budget constraints. On one hand, a hard budget constraint implies that bidders cannot pay more than a certain exogenously determined amount of money. On the other hand, senior management can set a soft budget constraint and inform the company's bidding team that it is not allowed to spend more than a certain amount of money. That is, under soft budget constraints, senior management may set aside a certain amount of money to be invested in acquiring spectrum. Soft budget may be updated during the auction when it turns out that a certain desired package cannot be obtained with the agreed upon budget. If soft budget constraints do affect bidding behavior, it is an interesting question to ask how these constraints can be optimally chosen or whether it is optimal not to have these constraints at all.[1] This paper considers, however, hard budget constraints only.

This paper points at different effects of hard budget constraints in a CCA under two alternative sets of assumptions concerning bidders' preferences. First, we consider "standard" preferences, where bidders only care about the spectrum they win and the price they have to pay for that spectrum. Second, we consider bidders having alternative preferences: in addition to their own surplus, they also care about the price other bidders pay for the spectrum they win. Under these alternative preferences, bidders have a spite motive and, *ceteris paribus*, prefer outcomes where rivals pay more for their winning allocation. Janssen and Karamychev (2014) argue that comments by the Austrian regulator RTR can be interpreted as saying that in the supplementary round bidders have made many bids on very large packages the bidders knew they were unlikely to win, and that these bids were effective in raising the prices other bidders had to pay. Janssen and Karamychev (2014) provide two arguments why real-world bidders in spectrum auctions are likely to engage in spiteful bidding, in addition to the evidence quoted from the Austrian regulator.[2]

We model the spite motive in a lexicographic way, i.e., a bidder always prefers outcomes with a larger intrinsic surplus (the value of the winning package minus the payment); rival payments only come into consideration to distinguish between outcomes with identical intrinsic surplus. This lexicographic preference ordering is an elegant way to select among the many equilibria of the supplementary round resulting in the same spectrum allocation, but different payments.[3]

Paradoxically (maybe), having a hard budget constraint does not mean that in a CCA, a bidder should avoid bidding above budget. In particular, a bidder may insert bids in the supplementary round that are not winning (and therefore do not affect the allocation). Bidders with a spite motive make these bids only to raise rivals' costs. Thus, we distinguish three ways in which a bidder can satisfy his budget constraint, depending on how much risk

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[1]See Burkett (2015) for endogenous budget constraints and Burkett (2014) for the principal's optimal choice of a budget in the single-unit case.

[2]Milgrom (2004) argues that bidders dislike different prices for similar packages. Raising other bidders' costs might come from these motives as well.

[3]If bidders have lexicographic preferences for raising rivals' costs, then the set of equilibria should coincide with the set of equilibria in which no bidder can raise other bidders' costs without decreasing his expected surplus.

he accepts that he has to pay more than his budget: (i) a conservative, (ii) a neutral, or (iii) a risky way. A *conservative* bidder never bids above budget. A *neutral* bidder only makes bids that are such that whatever feasible bids the other bidders make from that moment onwards in the clock or the supplementary phase, he will never pay more than his budget. As we will explain below, given the auction rules that apply, a bidder can calculate that certain bids cannot be winning. In a multi-bidder auction with many units being auctioned, this may require complicated combinatorial calculations. The difference between a conservative and a neutral bidder is that the first may not trust his own calculations (or the algorithm that his advisors are using). A *risky* bidder goes one step further than the first two types of bidders. A risky bidder has certain expectations of rivals' (future) bidding behavior, and given these expectations does not have to pay more than his budget, and these expectations are correct in equilibrium. As there may be multiple equilibria in a CCA, this type of bidder may have to pay more than budget, if his expectations turn out to be incorrect.

We obtain the following results for standard preferences. First, when considering the supplementary phase as a standard VCG mechanism, we show that under a budget constraint, the VCG mechanism does not have a weakly dominant strategy anymore. Instead, we characterize a range of bids on different packages that remain undominated. Bidders face the following trade-off: bidding full budget on all packages with a value larger than budget increases the chances of winning at least some package, but it may not be the most profitable package to win (given some bid strategy of others). This implies that even if bidders' behavior in the clock phase is such that the constraints on supplementary round bids are not binding, optimal bidding in the supplementary round may be nontrivial and dependent on bidders' expectations of rival bids.

We next analyze some aspects of clock phase bidding by means of two examples. A first example shows that the information during earlier clock rounds may be such that a bidder knows he can safely bid above budget without running the risk of winning that package and having to pay above budget. Bidding above budget may be beneficial as it relaxes the constraints on supplementary round bidding, allowing the bidder to bid true marginal values in that round. Bulow et al. (2009) define a bidder's exposure as the maximal amount a bidder has to pay if all bids become winning. Due to the pay-as-bid pricing rule, a bidder's exposure in the SMRA is simply equal to the sum of his bids. In the CCA, however, a bidder's exposure is the VCG price of the currently demanded package. The example shows that the exposure can be sufficiently different from the bid. In the example, the (positive) role of the clock phase is to provide bidders with information of their exposure, and thus, of the possibility of making bids without the risk of winning them. This role of the clock phase has so far been neglected. A second example shows that in CCAs where multiple bands are allocated the clock phase may actually last longer (depending on bidding behavior and how bidders react to the budget constraint) if bidders are budget-constrained. Moreover, in the clock phase bidders may face similar considerations as the ones we discussed above for the VCG mechanism indicating bidding in the clock phase under a budget constraint is strategically complex.

Finally, we show that in a CCA the spite motive interacts in a complicated way with budget constraints. In the context of an example, conservative bidders (those bidders without bids above budget) may have to pay more for identical packages than their risk-taking competitors pay. Ironically, conservative bidding is associated with the risk of having to pay more than competitors! In another example, budget constraints lead to multiple equilibria with a Hawk-Dove type flavor: aggressive, very risky, bidders perform well against neutral, or risky, but less aggressive bidders, but their bidding leads to payments above budget if all

bidders are aggressive.

Cramton (1995) and Salant (1997) highlight, among others, the importance of budget constraints in spectrum auctions. However, most academic papers on multi-unit auctions ignore budget constraints despite their practical importance. Che and Gale (1998) and Benoit and Krishna (2001) are early papers discussing single and multi-unit auctions with budget-constrained bidders respectively. If bidders are budget-constrained, the single-unit second-price auction has a weakly dominant strategy (e.g. Krishna 2010). On the contrary, the multi-unit version of the VCG auction does no longer have an equilibrium in weakly dominant strategies (see, e.g. Ausubel and Milgrom 2006 for an example where a bidder's optimal bid depends on the bid of a competitor). This already indicates the problems budget constraints impose on auction designers and bidders.

For the SMRA, Brusco and Lopomo (2008) show that private budget constraints may lead to strategic demand reduction and therefore to potentially inefficient outcomes. In a subsequent paper, Brusco and Lopomo (2009) analyze a simple model (two bidders, two units) of the SMRA with complementaries and known budget constraints. Without budget constraints there exists an efficient non-collusive equilibrium, but with budget constraints, the exposure problem might arise. In equilibrium, the bundle can be assigned to the bidder with lower budget and lower valuation for the bundle. A positive use of bidders' budget constraints is exemplified in Bulow et al. (2009). The authors describe a way to forecast relatively early in the auction the final revenue based on budget constraints in an SMRA. Moreover, they present a real-world example in which this information was successfully used in a high-stake spectrum auction. Ausubel and Milgrom (2002) introduce an ascending pay-as-bid auction. The pay-as-bid payment rule facilitates bidding under a budget constraint, since there is no uncertainty about the final price if a bid becomes winning.

Ausubel (2004) puts forward a dynamic version of the VCG mechanism and illustrates that bidding under a budget constraint might be easier and more efficient in the dynamic version than under the sealed-bid VCG mechanism. In an example much like our example 3, he shows that efficiency can be hard to obtain in the sealed-bid version, but relatively easy in his dynamic "clinching" auction. In the clinching auction, bidders learn their VCG price during the auction. If at least aggregate demand is revealed in every round, bidders know at which prices they clinched some goods. Therefore, they know the price they have to pay for their current clinches and can calculate the difference between budget and the price for current clinches. If this difference is above the current price, bidders can keep demanding truthfully. Ausubel's paper restricts attention to bidders with decreasing marginal values.[4] We focus on CCAs with possible complementaries across units where the exposure problem might arise. The CCA is another dynamic version of the sealed-bid VCG auction. Unlike Ausubel's (2004) auction, the CCA is a package auction that solves the exposure problem. In the CCA bidders do not directly learn parts of the allocation and final prices during the clock phase, but they can compute upper and lower bounds on final VCG prices. This information on bidders' exposure is provided through the activity rule and can be used to forecast a range of possible final prices. We show that if the forecast indicates that the final price cannot be above budget, neutral bidders (in the sense distinguished above) may find it optimal to bid above budget. Christian Kroemer and Goetzendorff (2015), Gretschko et al. (2016) and McKenzie and Fookes (2016) look at other aspects of bidding under budget

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[4] **?** show that in a setting very much like in Ausubel (2004) and with publicly known budget constraints, an "adaptive" version of Ausubels auction is the unique mechanism that is simultaneously pareto-optimal and incentive-compatible. However, if the budgets are private information, there is no incentive-compatible and pareto-optimal auction.

constraints in a CCA.

The rest of the paper is organized as follows. Section 2 determines the set of strategies that are not dominated in a VCG mechanism where bidders are budget-constrained and have standard preferences. Section 3 discusses the two examples illustrating the different optimal behaviors in the clock phase under a budget constraint. Section 4 discusses the complexities of combining spiteful bidding with a budget constraint. Section 5 concludes with a discussion. Proofs are in the Appendix.

2 Budget-Constrained Bidders in VCG

A well-known result for second-price auctions is that bidders have a weakly dominant strategy to bid their value. For one-unit auctions, this result has an analogue when bidders are budget-constrained: bidding the minimum of the value of the object and the budget is a weakly dominant strategy (see, *e.g.*, Krishna (2010, Proposition 4.2)). This Section shows that this result does not generalize to VCG auctions. Accordingly, bidding under a budget constraint is a non-trivial exercise in a multi-unit second-price auction.

To show which strategies are weakly dominated, and which cannot be eliminated as weakly dominated strategies, we use the following notation. Let there be K different types of objects to auction with n_k objects of type $k = 1, \ldots, K$. We use x to denote generic packages, and use Greek letter superscripts to refer to specific packages, *e.g.*, x^α. The set of all feasible packages is denoted by X, and the aggregate supply is denoted by $\overline{x} = (n_1, \ldots, n_K) \in X$. There are n bidders, and the intrinsic valuation of bidder i for any package x^α is denoted by $v_i^\alpha = v_i(x^\alpha)$. The set of all valuations of bidder i is denoted by $V_i = \{(x^\alpha, v_i^\alpha) : x^\alpha \in X\}$. Let $\Psi_i \subseteq X$ be a subset of packages that bidder i bids on in a VCG mechanism. Accordingly, let $\Phi_i = \{(x^\alpha, b_i^\alpha) : x^\alpha \in \Psi_i\}$ be the set of bidder i's bids in the VCG mechanism, where $b_i^\alpha = b_i(x^\alpha)$ is the monetary amount b_i^α that bidder i bids on package x^α. A feasible auction allocation is denoted by $A = (x_1^A, \ldots, x_n^A)$. Bidder i has a budget ω_i, which is assumed to be a hard budget restriction. When no confusion is possible we drop subscript i.

In the following proposition, we state which strategies (set of bids) are weakly dominated in the VCG mechanism, and which set of bids are not under the assumption that all bids are potentially pivotal (Milgrom 2004, p. 50).

Proposition 1. *Let all bids be potentially be pivotal and let $x^{\max}$ be the most valuable package of bidder i, i.e., $v_i(x_i^{\max}) \geq v_i(x)$ for all $x \in X$, and $v_i^{\max} = v_i(x^{\max})$ be the corresponding value. Then, a collection of VCG bids Φ_i is weakly dominated if, for some package x^α:*

1. $b_i^\alpha > \min(v_i^\alpha, \omega_i)$, *or*

2. $b_i^\alpha < \max\{\min(v_i^{\max}, \omega_i) + (v_i^\alpha - v_i^{\max}), 0\}$.

The set of undominated bids consists of:

1. $b_i^{\max} = \min(v_i^{\max}, \omega_i)$ on the most valuable package $x_i^{\max}$, and

2. $b_i^\alpha \in [\max\{\min(v_i^{\max}, \omega_i) + (v_i^\alpha - v_i^{\max}), 0\}, \min(v_i^\alpha, \omega_i)]$ on all other packages x^α.

The proposition can be relatively easily understood. Under a hard budget constraint it is never optimal to bid above value or above budget. In an optimal strategy a bidder bids the full budget on his most valuable package, and the bid difference between this bid and the bids on all other packages will *not* be larger than the difference in valuations. For these other packages, a bidder faces the trade-off between winning at least some package (in which case they will bid full budget on less valuable packages as well), or winning the most profitable package (in which case they will bid full budget minus the value difference on less valuable packages as well).

The following example, which also will be used in the next Section on strategic bidding under a budget constraint in the clock phase, illustrates the Proposition.

Example 1. Undominated strategies in the VCG auction with a budget constraint.

There are three bidders competing for one band in which four units are for sale. The set of feasible packages is, for simplicity, $X = \{1, 2, 3, 4\}$, and bidders' realized values are

Package x^α	(1)	(2)	(3)	(4)
Values of bidder 1: v_1^α	5.9	12	12	12
Values of bidder 2: v_2^α	5	9.5	10	10
Values of bidder 3: v_3^α	5	8	8	9

for one, two, three and four units respectively. Bidder 1 has a budget of $\omega_1 = 9$. Bidders do not know each other's valuation (private information scenario). In particular, bidder 1 knows that values of his rivals are either as stated above, or as summarized below.

Package x^α	(1)	(2)	(3)	(4)
Possible values of bidder 2: $\widehat{v}_2^\alpha$	3.5	10	10	10
Possible values of bidder 3: $\widehat{v}_3^\alpha$	4	9.5	10	10

We refer to the first set of valuations as "actual" and to the second set as "alternative". The winner determination problem is to find a feasible allocation $A = (x_1, x_2, x_3)$ that maximizes the function $b(A) = b_1(x_1) + b_2(x_2) + b_3(x_3)$ such that $x_1 + x_2 + x_3 \leq 4$.

Table 1 shows the values of the sum of all the bids for the two possible sets of valuations for different bids of the first bidder and truthful bidding of players 2 and 3, respectively. The bold entry indicates the highest sum of bids for the given bidding behaviors of the three bidders. If all bidders bid truthfully in the VCG auction, then the final auction allocation is $(2, 1, 1)$, and bidder 1's VCG price is

$$p_1^{VCG} = 17.5 - 10 = 7.5.$$

In the world of alternative preferences, bidder 1 still wins 2 units in the efficient (and final) allocation, but now he has to pay 9.5, which is above the budget of 9. Bidder 1 does not have a weakly dominant strategy. With actual preferences, he wants to win 2 units because the VCG prices is below budget. Therefore, it is better to submit a bid that makes it more likely that he wins 2 units. Preserving the true increase in utility from 1 to 2 units makes it more likely to win 2 units. If he bids

$$\Phi_1 = \{(1, 2.9), (2, 9), (3, 9), (4, 9)\}$$

Actual values of the other bidders						
Bids of Bidder 1	$b(2,2,0)$	$b(2,1,1)$	$b(1,2,1)$	$b(1,1,2)$	$b(2,0,2)$	$b(0,2,2)$
$b_1 = (5.9, 12, 12, 12)$	21.5	**22**	20.4	18.9	20	17.5
$b_1 = (5.9, 9, 9, 9)$	18.5	19	**20.4**	18.9	17	17.5
$b_1 = (2.9, 9, 9, 9)$	18.5	**19**	17.4	15.9	17	17.5
Alternative values of the other bidders						
Bids of Bidder 1	$b(2,2,0)$	$b(2,1,1)$	$b(1,2,1)$	$b(1,1,2)$	$b(2,0,2)$	$b(0,2,2)$
$b_1 = (5.9, 12, 12, 12)$	**22**	19.5	19.9	18.9	21.5	19.5
$b_1 = (5.9, 9, 9, 9)$	19	16.5	**19.9**	18.9	18.5	19.5
$b_1 = (2.9, 9, 9, 9)$	19	16.5	16.9	15.9	18.5	**19.5**

Notes: The rows present different bids of bidder 1. The columns indicate the respective sum of bids for an allocation if bidders 2 and 3 bid truthfully. The bold number indicates the maximal sum for a given bid of bidder 1 and truthful bids of the other two bidders.

Table 1: The impact of the bidding of a budget-constrained bidder on the final allocation in the VCG auction

the marginal increase from 1 unit to 2 units is higher, therefore it is more likely to win 2 units. On the other hand, he risks winning nothing. This can be seen in the lower panel of Table 1. Under actual preferences, he would win two units, but under the alternative preferences he would not win anything. A bid that makes it more likely that he wins 1 unit is

$$\Phi_1 = \{(1, 5.9), (2, 9), (3, 9), (4, 9)\}$$

If he submits this bid, the marginal bid on 1 is relatively large, therefore it is more likely that he will win 1 in the end. For both sets of preferences of rival bidders, he wins one unit, which is his optimal share as the VCG price for two units is above bidder 1's budget. ///

Thus, the example confirms that bidding under uncertainty and a budget constraint is a nontrivial task and the optimal bid depends on the beliefs about the other bidder's valuation and play.

Another important aspect of budget constraints in a VCG (and the CCA) is that they may affect the prices competitors pay, even if the budget itself is not binding in the sense that a bidder may not need to pay his full budget. In addition, the bidder that is most budget-constrained (often the smallest bidder), may be the one that in the end pays the most!

Example 2. Non-binding budget constraints may benefit competitors.

In this second example, there are three bidders competing for two bands with supply $\bar{x} = (6, 3)$ and the set of feasible packages is $X = \{(1, 1), (2, 1), (1, 2), (3, 1), (3, 2)\}$. Let all bidders have the following (symmetric) values:

$$V_i = \{((1, 1), 30.5), ((2, 1), 45), ((1, 2), 39.5), ((3, 1), 49), ((3, 2), 52)\}.$$

If bidders bid truthfully and bid their values on all packages, $\Phi_i = V_i$, then it is easy to see that the auction allocation is $A = ((2,1),(2,1),(2,1))$ with an auction price of

$$p_i^{VCG}(2,1) = (b(3,2) + b(3,1)) - (b(2,1) + b(2,1)) = 11.$$

Each bidder gets a surplus of 34.

Suppose now that bidder 1 has a budget of $\omega_1 = 45$ and that he uses this budget so as to maximize his chances of winning the most profitable package without paying more than his budget. He will do so by bidding

$$\Phi_1 = \{((1,1),30.5),((2,1),45),((1,2),39.5),((3,1),45),((3,2),45)\}.$$

Note that the budget is much higher than what he has to pay if all bidders bid their value. If the other two bidders continue to bid their values, $\Phi_2 = V_2$ and $\Phi_3 = V_3$, then the winning allocation is unaffected and so is the price the budget-constrained bidder has to pay. However, the two other bidders gain from the budget-constraint of their competitor and now only pay 7, instead of 11, since

$$p_2^{VCG}(2,1) = (b_3(3,2) + b_1(3,1)) - (b_3(2,1) + b_1(2,1)) = 7.$$

Thus, the only budget-constrained bidder in this example is the one that pays the most even though all bidders acquire identical packages. ///

The next Section adds a clock phase to these two examples to show that sometimes the clock phase may provide bidders with information that allow them to infer that the VCG price is below budget so that they may actually bid above budget (which may restore efficiency).

3 Budget-Constraints in the clock of the CCA

This Section concentrates on some aspects of strategic bidding under a budget constraint that explicitly involve the clock phase of the CCA. First, we show that the outcome of the CCA can be efficient, whereas the outcome of the VCG auction is not necessarily efficient. This may be the case when bidders are willing to bid above budget as long as they can compute that their exposure is not more than their budget (this is what we have called neutral bidding). The VCG mechanism does not provide bidders with information which bids of the other bidders are feasible, and therefore it may be the case that bidders have to pay more than their budget if they make some bids above budget. The dynamic nature of the clock phase of the CCA, paired with the activity rule that links the clock phase to the supplementary phase, allows bidders to compute upper bounds on the final VCG price if they themselves choose to make certain bids. This is sometimes called a bidder's exposure. If the exposure is below budget, bidders may safely bid above budget without running the risk of having to pay more than their budget.

Second, if during the clock phase bidders infer their exposure is above their budget constraint, similar considerations to the ones we have identified in the previous Section for the VCG mechanism apply during the clock. A bidder may reduce demand to 0 in an attempt maximize his chances to get his most preferred outcome, but may also bid on the most profitable package for which it still can assure it will not pay above budget to maximize the chance to get at least some package. Moreover, we show that in CCAs where multiple

bands are allocated the clock may actually last longer (depending on bidding behavior and how bidders react to the budget constraint) if budget-constrained bidders choose the latter option.

In the examples we use the following notation: The clock round prices are denoted by p^t, the clock round demand by D_i^t and the final auction price bidder i pays for package x^α by $p_i^{VCG}(x^\alpha)$.

3.1 An efficiency restoring role of the clock phase

The activity rules of a CCA translate bidders' clock demand to constraints on the admissible bidding function in the supplementary phase. Bidders can use the information that is revealed in the clock phase to compute upper (and lower) bounds on the other bidders' supplementary bidding function. This information allows them to forecast the maximal VCG price, *i.e.*, their exposure. Thus, depending on the development of the clock phase and the monetary value of the budget, bidders may learn that even if they bid above budget in the clock phase, and subsequently in the supplementary phase, they never have to pay above budget. The applicability of this observation crucially depends on the activity rules and the information disclosure policy.[5] The more information on the other bidder's clock demand is revealed, the better budget-constrained bidders are able to forecast future final prices. The next example shows how this may work.

Example 3. The possible efficiency restoring role of the clock phase.

We reconsider example 1, but now introduce a clock phase. For simplicity, we assume that bidders learn the individual demands after each clock round. Table 2 summarizes the demands at the given price. The clock starts at a price of 1. Due to excess demand the price increases up to 4. At the price $p = 4$, bidder 1 observes that bidder 3 reduced demand to one. Since there is still excess demand, the price will be increased to 5 in the next clock round. Under truthful bidding, bidder 1 would demand two units at this price. In this case he has to bid at least 10, which is above budget, for two units in the supplementary phase. He wants to bid above budget only if he knows that the final VCG price is below budget. Bidder 1 can infer from the observed history that in all admissible continuations of the clock, the final VCG price is below budget.

p	$D_1(p)$	$D_2(p)$	$D_3(p)$
1	2	2	2
2	2	2	2
3	2	2	2
4	2	2	1

Table 2: Observed bids in the clock phase

If bidder 1 demands two units at a price of 5, then the clock can end at $p = 5$ with five possible final clock allocations that are consistent with the observed history. First, the clock can end with market clearing, in which case the final clock round demands are either

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[5]The activity rules used in this Section are the final and the relative cap.

| Possible clock demand histories | | | | | Constraints on $b_i(1)$ | Constraints on $b_i(2)$ |
$D_i(1)$	$D_i(2)$	$D_i(3)$	$D_i(4)$	$D_i(5)$		
2	2	2	2	2	$b_i(1) \leq b_i(2) - 5$	$10 \leq b_i(2)$
2	2	2	2	1	$5 \leq b_i(1)$	$8 \leq b_i(2) \leq b_i(1) + 5$
2	2	2	2	0	$b_i(1) \leq 5$	$8 \leq b_i(2) \leq 10$
2	2	2	1	1	$5 \leq b_i(1)$	$6 \leq b_i(2) \leq b_i(1) + 4$
2	2	2	1	0	$4 \leq b_i(1) \leq 5$	$6 \leq b_i(2) \leq b_i(1) + 4$

Table 3: Transformation of clock demands into constraints on the supplementary bidding function

$(2, 2, 0)$ *or* $(2, 1, 1)$. *Second, it can end with excess supply, in which case the final clock demands are either* $(2, 1, 0)$, *or* $(2, 0, 1)$, *or* $(2, 0, 0)$. *The clock continues only if* $(2, 2, 1)$ *is demanded in round 5.*

Table 3 summarizes the constraints on the supplementary bidding function for all possible clock demands that are consistent with the observed demands in Table 2 and the clock ending at $p = 5$. *The constraint on* $b_i(3)$ *and* $b_i(4)$ *are the same for all bidders. In the supplementary phase, it must be true that* $b_i(3) \leq b_i(2) + 1$ *and* $b_i(4) \leq b_i(2) + 2$.

Bidder 1 can now compute the possible final VCG prices if the clock ends at $p = 5$ *by him demanding two units. Similarly, bidder 1 can compute possible final VCG prices if he demands two units at* $p = 5$ *and the clock continues and he bids such that he respects his budget constraints at* $p > 5$. *In this case it must be that the demand equals* $(2, 2, 1)$ *at a price of 5. The detailed calculations in the Appendix show that in either case, bidder 1 will not have to pay more than his budget. Intuitively, if the final allocation is* $(2, 1, 1)$, *then the other bidders can make him pay at most 9, since bidders 2 and 3 cannot raise their bids on two units more than 5 and 4 respectively.*

If the clock finishes at $p = 5$, *by bidding for two units at the final clock price bidder 1 is able to bid his true marginal values on all packages in the supplementary round. Thus, using the VCG pricing rule he will always acquire the bundle with the highest intrinsic value, and he does not want to deviate from the truthful bidding strategy.* ///

The example depends on the activity rule and the information policy of the auction. The analysis was performed under the assumption that bidders learn the other bidders' past demand after every clock round. In the current example, revelation of aggregate demand would suffice, however, to obtain the same result. In more complicated auctions with more bidders and multiple bands, the more information is revealed about demand the better bidders are able to compute VCG prices (and their exposure) accurately. If no information about demand is revealed (like in the beginning of the clock phase in the 2013 Austrian auction), bidders cannot compute their exposure.

The auction designer faces a trade-off in the choice of informational policy. Revealing more information about demand can foster collusion or spiteful behavior (as in Janssen and Karamychev (2014) and Levin and Skrzypacz (2014)), but it can also enable efficient outcomes when bidders face budget constraints.

3.2 Budget constraints may prolong the clock

If bidders cannot infer that their exposure is smaller than their budget, budget-constrained bidders have to adjust their clock phase bidding. The next example of a CCA with multiple bands demonstrates that bidders have to make strategically difficult decisions during the clock phase. By significantly reducing demand or by dropping out of the clock phase altogether, a bidder may maximize his chances to get his most preferred outcome. On the other hand, such a decision also increases the chance of not winning anything and, from this perspective, it may be better to simply bid on the most profitable package for which the calculated exposure is below budget. With multiple bands, bidders may adjust their bidding behavior in such a way that the clock may actually last longer if budget-constrained bidders choose to bid on the most profitable package for which the calculated exposure is below budget.

Example 4. Budget constraints may extend the clock if a bidder's exposure is above budget.

We reconsider example 2 with a clock phase. Suppose that the eligibility points for the first band equal 1 and that they equal 2 for the second band. We assume that the CCA begins with reserve prices $p^1 = (10, 1)$, and price increments in both bands are equal to one. The following table represents the clock phase development. If bidders bid truthfully in the clock phase their behavior is given by the next table. It is clear that round $t = 5$ is the final clock round.

Round, t	Prices, p^t	D_1^t	D_2^t	D_3^t
1	$(10, 1)$	$(1, 2)$	$(1, 2)$	$(1, 2)$
2	$(10, 2)$	$(1, 2)$	$(1, 2)$	$(1, 2)$
3	$(10, 3)$	$(1, 2)$	$(1, 2)$	$(1, 2)$
4	$(10, 4)$	$(1, 2)$	$(1, 2)$	$(1, 2)$
5	$(10, 5)$	$(2, 1)$	$(2, 1)$	$(2, 1)$

If bidders also bid truthfully in the supplementary round, they bid their values on all packages, $\Phi_i = V_i$, so that, like in the example 2, all bidders pay 11 for the package $(2, 1)$, and their surplus equals 34.

Suppose now that bidder 1 has a budget of $\omega_1 = 23$. The exposure of bidding $D_i^5 = (2, 1)$ in round $t = 5$ is larger than the available budget. If the auction would end at that round, the others could maximally raise their bid $b(3, 2)$ to $b(2, 1) + 15$ and their bid $b(3, 1)$ to $b(2, 1) + 10$ so that together the other bidders can raise the price bidder 1 has to pay for obtaining package $(2, 1)$ to 25. As a result of the budget constraint, bidding $D_i^5 = (2, 1)$ in round $t = 5$ is not feasible for him. Among the (still feasible) packages, $(1, 1)$ and $(1, 2)$, package $(1, 2)$ is the most profitable one at prices $p^5 = (10, 5)$ so that one may assume bidder 1 bids $D_i^5 = (1, 2)$. If this happens, the price for band 2 keeps increasing until round $t = 7$, when bidder 1 switches to the package $(1, 1)$ and the clock stops consequently. The following table represents the clock phase development for rounds $t = 5, \ldots, 7$ if bidder 1 has a budget constraint of $\omega_1 = 23$.

Round t	Prices p^t	D_1^t	D_2^t	D_3^t
5	$(10,5)$	$(1,2)$	$(2,1)$	$(2,1)$
6	$(10,6)$	$(1,2)$	$(2,1)$	$(2,1)$
7	$(10,7)$	$(1,1)$	$(2,1)$	$(2,1)$

The relative cap rule imposes the following restrictions on the supplementary round bids of bidder $i = 1$:

$$b_1(2,1) \leq b_1(1,1) + 10,\ b_1(1,2) \leq b_1(1,1) + 7,$$
$$b_1(3,1) \leq b_1(1,1) + 20,\ b_1(3,2) \leq b_1(1,2) + 20.$$

If bidders 2 and 3 bid truthfully in the supplementary round, and bidder 1 bids according to his budget:

$$b_1(1,1) \in [17,23],\ and\ b_1(2,1) = b_1(1,2) = b_1(3,1) = b_1(3,2) = 23,$$

bidder 1 wins $(1,1)$ at price

$$p_1^{VCG}(1,1) = (b_2(3,2) + b_3(3,1)) - (b_2(2,1) + b_3(3,1)) = 52 - 45 = 7$$

and obtains a surplus of 23.5. In this case, bidders $i = 2, 3$ win $(2,1)$ and $(3,1)$ at prices

$$p_2^{VCG}(2,1) = (b_3(3,2) + b_1(3,1)) - (b_3(3,1) + b_1(1,1)) = 26 - b_1(1,1)$$
$$p_3^{VCG}(3,1) = (b_2(3,2) + b_1(3,1)) - (b_2(2,1) + b_1(1,1)) = 30 - b_1(1,1)$$

with surplus $(19 + b_1(1,1))$ from both packages.

Alternatively, having observed that bidders 2 and 3 have switched to $(2,1)$ in round $t = 5$, bidder $i = 1$ can drop to $(0,0)$ in round $t = 6$. The following table represents the clock phase development for rounds $t = 5, 6$.

Round t	Prices p^t	D_1^t	D_2^t	D_3^t
5	$(10,5)$	$(1,2)$	$(2,1)$	$(2,1)$
6	$(10,6)$	$(0,0)$	$(2,1)$	$(2,1)$

The relative cap rule imposes the following restrictions on the supplementary round bids of bidder 1:

$$b_1(2,1) \leq 26,\ b_1(1,2) \leq 22,\ b_1(3,1) \leq 36,\ b_1(3,2) \leq b_1(1,2) + 20.$$

If bidders $i = 2, 3$ bid truthfully in the supplementary round, and bidder 1 bids $b_1(2,1) = 23$, he wins $(2,1)$ at price

$$p_1^{VCG}(2,1) = (b_i(3,2) + b_j(3,1)) - (b_i(2,1) + b_j(2,1)) = 11$$

as in the case of no budget. Therefore, bidder 1 has a chance to win $(2,1)$ by dropping out of the clock phase altogether. He pays a price smaller than his budget and obtains a surplus that is larger than the surplus that he obtains by bidding $(1,2)$ in round 6 and $(1,1)$ in round 7. ///

Example 4 shows that budget-constrained bidders may face strategically difficult decisions in the clock phase of the CCA. Depending on the bidding behavior of their competitors, it may be optimal to drop out of the clock phase or to continue bidding for packages whose exposure is within the budget limit. Depending on how a bidder resolves this dilemma, we have illustrated by means of an example that the clock phase may, paradoxically, be extended if bidders are budget-constrained.

4 Budget-Constrained Bidders Under Spite Motive

In this Section, we extend the analysis to include bidders with a preference to raise rivals' costs. As we explained in the Introduction, there are good reasons why in spectrum auctions, bidders are not indifferent across auction outcomes that differ in terms of what competitors pay for their spectrum. As bidders' payments in a CCA depend on competitors' bids, this implies that each bidder may want to investigate to what extent they are able to raise rivals' costs. We will say that bidders bid to raise rivals' cost if the bid difference between two packages is larger than the value difference between the same two packages. In general, bidders do not want to bid in such a way that their bids on packages that were intended to raise rivals' costs end up winning and that the marginal payment they have to make for winning that package is larger than the value difference. Under a budget constraint, there is an alternative concern, namely that bidders do not want to pay more than their budget.

In the Introduction, we have also argued that there are different ways to satisfy the budget constraint. First, bidders may bid in such a way that independent of the behavior of their competitors, they will never have to pay more than their budget. Second, there is an equilibrium interpretation that allows bidders to bid in such a way that given correct expectations of their competitors' behavior, bidders bid in such a way that they do not have to pay above budget. In this Section, we will exemplify these notions by considering an example where all bidders have a budget constraint.

The motive to raise rivals' costs is modeled in the following lexicographic way. By the end of the supplementary round, any bidder has bid on a set of packages. A bidder either wins one of these packages and pays the opportunity cost of winning this package imposed on others, or he does not win. A bidder's intrinsic pay-off of bidding equals the standard surplus (value - payment). If, for a fixed strategy profile of other bidders, the intrinsic pay-off of two strategies is identical, bidders prefer the strategy that raises the sum of rivals' payments most. Thus, we can say a strategy σ dominates another strategy σ' if:

1. For any bids of the other bidders, σ never results in a lower intrinsic pay-off, or in the same intrinsic pay-off and a lower sum of rivals' payments than σ'; and

2. For some of the others' bids, either σ results in a larger intrinsic pay-off, or it results in the same intrinsic pay-off but raises the sum of rivals' payments, as compared to σ'.

In previous Sections, we have seen that under some conditions bidders can place bids above budgets on certain packages and win those packages at prices that are below the budgets. Without making such bids, bidders could not have won those packages. Under the spite motive, bidders have yet another reason to bid above budgets, namely to raise prices that other bidders pay. In other words, bidders may find it optimal to place bids above budgets on certain packages without an intention of winning them.

Example 5. Raising auction prices by bidding above budget.

We reconsider the example 4. Let all three bidders have budget $\omega_i = 23$. Under the assumption that bidders bid according to values satisfying the budget constraint, the clock phase will develop as in the next table.

Round, t	Prices, p^t	D_1^t	D_2^t	D_3^t
1-6	$(10, t)$	$(1, 2)$	$(1, 2)$	$(1, 2)$
7	$(10, 7)$	$(1, 1)$	$(1, 1)$	$(1, 1)$

Relative bid caps and feasibility restrictions on the supplementary round bids are as follows

$$
\begin{aligned}
17 &\leq b_1(1, 1) \\
b_1(2, 1) &\leq b_1(1, 1) + 10 \\
22 &\leq b_1(1, 2) \leq b_1(1, 1) + 7 \\
b_1(3, 1) &\leq b_1(1, 1) + 20 \\
b_1(3, 2) &\leq b_1(1, 2) + 20.
\end{aligned}
$$

The maximal safe bid $\widehat{b}$ for a package is the maximal bid such that the package never becomes winning. The table

Package x^α	$(1, 1)$	$(2, 1)$	$(1, 2)$	$(3, 1)$	$(3, 2)$
Maximal safe bids $\widehat{b}_1^\alpha$	23	23	30	23	40

specifies the maximal safe bids of bidder 1. Any feasible bid on $(1, 2)$ is never winning. Indeed, $b_1(1, 2)$ can only be winning if

$$b_1(1, 2) \geq b_1(1, 1) + b_i(1, 1) \geq b_1(1, 1) + 17,$$

but the feasibility constraint on $b_1(1, 2)$ says that it should not be larger than $b_1(1, 1) + 7$. Thus, bidding above budget on certain packages may be without risk of actually winning the package.

On other packages, it is not difficult to see that by exceeding the maximal safe bid, bidder 1 runs a risk of winning the package and paying above budget. For example, if bidder 1 increases $b_1(3, 1)$ by $x > 0$ above 23 the auction allocation may be $A = ((3, 1), (0, 0), (3, 2))$, and the price bidder 1 pays equals

$$p_1^{VCG}(3, 1) = (b_2(3, 1) + b_3(3, 2)) - b_3(3, 2) = 23 + \frac{1}{2}x > \omega_i$$

if other bidders bid as below.

Package x^α	$(1, 1)$	$(2, 1)$	$(1, 2)$	$(3, 1)$	$(3, 2)$
Bids of bidder 1, b_1^α	23	23	30	$23 + x$	40
Bids of bidder 2, b_2^α	$17 + \frac{1}{2}x$		22	$23 + \frac{1}{2}x$	
Bids of bidder 3, b_3^α	17		24	36	43

Similarly, bidding $b_1(3,2) = 40 + x$ brings a risk of winning $(3,2)$ if others bid as in the next table.

Package x^α	$(1,1)$	$(2,1)$	$(1,2)$	$(3,1)$	$(3,2)$
Bids of bidder 1, b_1^α	23	23	30	23	$40 + x$
Bids of bidder 2, b_2^α	17		22	29	$40 + \frac{1}{2}x$
Bids of bidder 3, b_3^α	17		22	29	$40 + \frac{1}{2}x$

In this case, the price bidder 1 would pay equals

$$p_1^{VCG}(3,2) = (b_2(3,1) + b_3(3,2)) - b_2(3,1) = 40 + \frac{1}{2}x > \omega_i.$$

Suppose all three bidders bid the maximal safe bids (to be more precise, let them bid $b_i(2,1) = 23$, and bids $b_i(1,1)$ and $b_i(3,1)$ be marginally lower than 23). Then, the allocation is $A = ((2,1),(2,1),(2,1))$ with price $p_1^{VCG}(2,1) = 17$. The high bid on package $(3,2)$ does not run the risk of being a winning package, but is actually very effective in raising rivals' costs as it is used to determine the prices other bidders pay. The neutral bidder we have identified in the Introduction may make such a bid and does not risk winning it. If all bidders would be conservative and do not make bids above 23, then all would win the package $(2,1)$ at price $p_1^{VCG}(2,1) = 0$. One neutral bidder could raise the prices of the others to 17 without raising his own price. ///

A problem with the assumed bidding behavior is that it is not an equilibrium. In particular, one of the bidders, say bidder 1, may increase his bids $b_1(3,1)$ and $b_1(3,2)$ beyond maximal safe bid in order to raise his rivals' costs. Our final example shows how equilibrium bidding in the supplementary phase may involve all bidders making risky bids, but one of them being more aggressive than the others. The equilibrium is asymmetric and has a Hawk-Dove flavor where the more aggressive "Hawk" bidder pays the lowest prices. The equilibrium presented satisfies the budget constraint "in equilibrium", but all bidders run the risk of having to pay more than their budgets if they fail to coordinate on this specific equilibrium.

Example 6. Hawk-Dove equilibria of the supplementary phase.
We continue the previous example. Let $x \in [0,4]$, and bidders bid in the supplementary round

$$\Phi_1 = \{((1,1),17),((2,1),23),((1,2),24),((3,1),29),((3,2),40+x)\}$$
$$\Phi_i = \{((1,1),17),((2,1),23),((1,2),24),((3,1),29-x),((3,2),40)\},$$

where $i = 2,3$. Here again, bids on packages $(3,1)$ and $(3,2)$ are marginally lower than the reported amounts and, therefore, not winning. Alternatively, there is a tie-breaking rule in place such that if multiple allocations maximize the total sum of bids, the allocation with the maximal number of winning bidders is chosen. In both cases, the winning allocation is $A = ((2,1),(2,1),(2,1))$ with prices

$$p_1^{VCG}(2,1) = (b_2(3,1) + b_3(3,2)) - (b_2(2,1) + b_3(2,1)) = 23 - x$$
$$p_i^{VCG}(2,1) = (b_1(3,1) + b_j(3,2)) - (b_1(2,1) + b_j(2,1)) = 23.$$

Bidders get surpluses of $22 + x$, 22, and 22 correspondingly.

For any $x \in [0, 4]$, this bidding behavior constitutes an equilibrium of the supplementary round. By increasing his bid on $(1, 1)$, bidder i wins $(1, 1)$ at price 17 with surplus 13.5. By increasing his bid on $(3, 1)$, bidder $i = 2, 3$ wins $(3, 1)$ at price $29 - x$ with surplus $20 + x$, but the price is well above budget. Finally, by increasing his bid on $(3, 2)$, bidder $i = 2, 3$ wins $(3, 2)$ at price 40 with surplus 12. Similar arguments establish that bidder 1 cannot benefit from deviating. Thus, no player can deviate profitably.

Note that there is a continuum of these asymmetric equilibria. In each of these equilibira, bidder 1, the Hawk, bids more aggressively than bidders $i = 2, 3$, the Doves. Being aggressive pays off, as the more aggressive bidder pays less for identical spectrum. Thus, there is not only a coordination problem as to which value of x to stick to, but also who of the three bidders may play the role of the more aggressive bidder. If bidders do not coordinate and at least two of them play the aggressive strategy of bidder 1, then the winning allocation is that two bidders win $(3, 1)$ and $(3, 2)$, respectively, while each has to pay an amount that is much larger than their budget. ///

The examples illustrates the different ways to bid in a CCA under a budget constraint. First, conservative bidders who make all their bids below their budgets may pay more than others when they do not express a higher bid for larger packages. Second, a neutral bidder calculates the maximal bids he can make on different packages in order to be certain not to win them and bids accordingly to raise rival's cost. In many instances, the bidding strategies of neutral bidders do not form an equilibrium, however, and risky bidders may want to increase their bids further when these higher bids will not be winning given (rational) expectations of what others will bid. The CCA encourages bidders to be aggressive in case others are restrained as aggressive bidders may well pay considerably less for identical packages.

5   Conclusion

Mainly by analyzing illustrative examples, this paper has considered the implications of bidders in a combinatorial clock auction to be budget constrained. As the last phase of the CCA is a kind of Vickrey-Clark-Groves (VCG) mechanism, we first have considered the implications of a budget constraint in the standard VCG mechanism. We have shown the range of bids that are and are not weakly dominated. In general, a bidder faces a trade-off between bidding budget on many possible packages to acquire at least one of them or to differentiate the bids on different packages by their value difference to acquire the most profitable one. Trying to get the most profitable package runs the risk of not winning any package as all bids are too low to be selected by the winner determination algorithm. We have also shown that by bidding budget on many possible packages, a budget constrained bidder may pay more for an identical package than his unconstrained competitors.

We then considered the clock phase of the CCA and argued that it may be beneficial to provide bidders with detailed information about the bidding behavior of their competitors. With this information, bidders are able to calculate their exposure at the beginning of each clock round. Knowing their exposure is below budget, this information allows them to bid above budget. In an example, we show that this may improve the efficiency of the auction. Compared to the VCG mechanism, this also provides a positive, unexplored role for the clock phase. We also show that if a bidder's exposure is above budget, then bidders face

the same trade-off in the clock phase of the CCA as in the VCG mechanism (or in the supplementary round of the CCA).

Finally, we show that if, *ceteris paribus*, bidders have an incentive to raise rivals' costs (what is also called a *spite motive*), they always want to increase their bids on non-winning packages to the maximal extent possible. We have identified that in a CCA this does not imply bidders do not bid above budget. Budget-constrained bidders have two alternative ways they can satisfy their budget-constraint. First, bidders are in the position to calculate for each package the maximal bid they can make such that it cannot be a winning bid *no matter what their competitors are bidding*. Spiteful bidders want to make these bids in order to raise rivals' costs. Second, bidders may make bids such that in equilibrium no bidder pays an amount that is larger than their budget. In this case, bidders have specific and correct expectations concerning the bidding behavior of their competitors and given these expectations their bids are optimal and such that their final payment stays within their budget. Depending on the specific environment, the second interpretation of bidding under a budget constraint allows bidders to be (much) more aggressive than the first one and may lead to asymmetric equilibria of a Hawk-Dove nature, where the aggressive Hawk is much better off than the more peaceful Doves (given that they coordinate in the prescribed way). If coordination does not come about and multiple bidders bid according to the Hawkish strategy profile, they win large packages and have to pay prices above budget.

To sum up, the clock can be beneficial to the bidders because they can use the demand of the other bidders to compute upper and lower bounds of the final VCG price for a package. This information can be used eventually to place bids above budget (but below value) in order to win the most desired package. This can restore efficiency. However, the information released in the clock can also be used to raise rivals' costs.

6   Appendix

Proof of Proposition 1.
We omit the subscript i. Any bid above the value, $b^\alpha > v^\alpha$, is dominated by $b^\alpha = v^\alpha$. Next, any bid above budget, $b^\alpha > \omega$, is dominated by $b^\alpha = \omega$. This proves part (1) of the proposition. In order to prove part (2), we define $z^\alpha = \min(v^{\max}, \omega) + (v^\alpha - v^{\max}) - b^\alpha$, and $Z = \{x^\alpha : \forall x^\beta \in \Psi : z^\alpha \geq z^\beta\}$. In other word, Z is a subset of packages that generate the largest surplus had the bidder win them at VCG prices equal to their bids b^α. Consider an alternative set of bids $\widetilde{\Phi} = \{\widetilde{b}^\alpha : x^\alpha \in \Psi\}$ where $\widetilde{b}^\alpha = b^\alpha$ if $x^\alpha \notin Z$ and $\widetilde{b}^\alpha = b^\alpha + \varepsilon$ if $x^\alpha \in Z$. In other words, we raise bids on all packages from Z by a small amount $\varepsilon > 0$. It is easy to verify that $\widetilde{\Phi}$ dominates Φ.

Detailed calculations for Example 3
Suppose first that the clock phase ends at $p = 5$ and bidder 1 has demanded two units. In the determination of the final allocation and the VCG prices, no bidder's bid on 4 can ever play a role. To see this, note that

$$b_i(4) \leq b_i(2) + 2 < b_i(2) + 6 \leq b_i(2) + b_j(2),$$

that is, the bid on 4 is always smaller than the bid on 2 by the same bidder and the bid on 2 by another bidder. If the clock ends with demands $(2, 2, 0)$, then this is the final allocation by the final cap rule. Since the bid on 4 plays no role in the determination of VCG prices,

there are in principle three possible ways to construct the VCG price for bidder 1. However, only the cases

$$p_1^{VCG} = b_2(3) + b_3(1) - b_2(2)$$
$$p_1^{VCG} = b_2(2) + b_3(2) - b_2(2)$$

are relevant, since

$$b_2(2) + b_3(2) \geq b_2(2) - 5 + b_3(2) + 1 \geq b_2(1) + b_3(3).$$

In the first case, the VCG price is at most $p_1^{VCG} \leq 6$. This constraint comes from the fact that the bid on 3 can be at most 1 larger than the bid on 2 and from the last line in Table 3. In the second case, $p_1^{VCG} = b_3(2) \leq 9$, which is equal to the budget.

If the clock ends with $(2, 1, 1)$, then the final clock allocation is the final allocation. The possible VCG prices are

$$p_1^{VCG} = b_2(3) + b_3(1) - b_2(1) - b_3(1) \leq b_2(2) + 1 - b_2(1) \leq 6$$
$$p_1^{VCG} = b_2(2) + b_3(2) - b_2(1) - b_3(1) \leq b_2(1) + 5 - b_2(1) + b_3(1) + 4 - b_3(1) = 9.$$

If the clock ends with excess supply and if bidder 1 bids $b_1 = (5.9, 10, 10, 10)$, then in any final allocation in which bidder 1 gets 2 units, he does not pay more than budget. If the final allocation is $(2, 2, 0)$ or $(2, 1, 1)$, the same considerations as above apply. If the final allocation is $(2, 0, 2)$, then

$$p_1^{VCG} = b_2(2) + b_3(2) - b_3(2) = b_2(2) \leq 10.$$

But if $b_2(2) \geq \omega_1$, then $(2, 0, 2)$ is no longer implemented by the auctioneer since

$$b_1(1) + b_2(2) + b_3(1) \geq 5.9 + 9 + b_3(1) \geq 10 + b_3(1) + 4 \geq b_1(2) + b_2(2)$$

is true. Moreover, bidder 1 never gets 3 units since

$$b_1(3) + b_j(1) \leq b_1(2) + b_j(1) \leq b_1(2) + b_j(1) + b_{5-j}(1),$$

for $j = 2, 3$. Even if $b_2(2) = 0$, it is true that $b(1, 2, 1) > b(3, 0, 1)$.

Consider then the case where bidder 1 demands two units at $p = 5$ and the clock continues, so that the demand at $p = 5$ must be $(2, 2, 1)$. If the clock ends at some higher price $p > 5$, bidder 1 also never has to pay more than budget. If the clock continues, bidder 1 can drop demand to 0 at $p = 6$ and submit the supplementary bid $b_1 = (5.9, 10, 10, 10)$. Since he never has to pay more than his bid, by doing so the only possibility in which he has to pay more than budget is if he wins two units. But in no final allocation in which he wins two units, he has to pay more than budget. First, note that $(2, 2, 0)$ is never the final allocation since

$$b_1(1) + b_2(2) + b_3(1) \geq 5.9 + b_2(2) + 5 > b_1(2) + b_2(2) = 10 + b_2(2).$$

Second, also $(2, 0, 2)$ is never winning since

$$b_1(1) + b_2(2) + b_3(1) \geq 5.9 + 10 + b_3(1) \geq 10 + 4 + b_3(1) \geq b_1(2) + b_3(2).$$

Third, $(2, 1, 1)$ is winning if

$$b_1(1) + b_2(2) + b_3(1) \leq b_1(2) + b_2(1) + b_3(1)$$
$$b_2(2) \leq 4.1 + b_2(1).$$

In this case, the VCG price is either

$$p_1^{VCG} = b_2(2) + b_3(2) - b_2(1) - b_3(1) \leq 4 + 4.1 < 9,$$

or

$$p_1^{VCG} = b_2(3) + b_3(1) - b_2(1) - b_3(1) \leq 5.1 < 9,$$

which is less than budget in both cases.

As a result, bidder 1 can safely demand two units at a price of 5 in the clock phase. He knows that the final VCG price is never above budget.

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