The Combinatorial Multi-Round Ascending Auction[*]

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Abstract

The Combinatorial Multi-Round Ascending Auction (CMRA) is a new auction format that has already been used in several recent European spectrum auctions. We characterize ex-post equilibria that feature auction-specific forms of truthful bidding, demand expansion, and demand reduction for settings where bidders have either decreasing or non-decreasing marginal values. In particular, we show that the truthtelling equilibrium is fragile to small asymmetries in the bidders' caps. On the other hand, if bidders are sufficiently symmetric, the CMRA is vulnerable to risk-free collusion. We propose an alternative activity rule that prevents such collusive strategies but keeps the other equilibria intact. We discuss to what extent our theory is consistent with outcomes in Danish spectrum auctions and how our predictions can be tested using bidding data.

Keywords: combinatorial auctions, spectrum auctions, demand reduction, demand expansion, collusion, complementarities.

JEL: D44, D47, L41, L96.

1 Introduction

Auction sales of radio spectrum have served as a powerful engine for auction format innovation. These high-stake sales often involve heterogeneous spectrum bands and require flexible and robust auction formats. For example, the Simultaneous Multi-Round Auction (SMRA), the Combinatorial Clock Auction (CCA), and the Deferred Acceptance Auction, all trace their origins to spectrum auctions. Many dynamic auctions work well—they find efficient outcomes, give bidders decent incentives to bid truthfully, and are fairly robust to collusion—when the goods for sale are substitutes (Kelso Jr and Crawford, 1982; Milgrom, 2000). However, in the presence of complements—which is very common in spectrum allocation—auction design becomes substantially more difficult. To help bidders avoid ending up with undesirable combinations of goods, many *combinatorial* auction formats, such as the CCA and the sealed-bid combinatorial auction, allow bidders to express preferences over packages. In its 2016 spectrum auction, the Danish Energy Agency (DEA) pioneered the *Combinatorial Multi-Round Ascending Auction* (CMRA) format developed by the UK-based consultancy DotEcon Ltd. to sell several spectrum blocks in the 1800 MHz band (DotEcon, 2016). Since then, the CMRA has been used twice again by the DEA and once by the Norwegian regulator (Nkom).

The CMRA works as follows. There is a price clock for each good. In each clock round, bidders can report a *headline demand* for a single package which is priced linearly at clock prices. The novel feature of the CMRA is that in each clock round, bidders can also submit *additional* (package) *bids* at or below the clock prices (and subject to an activity rule). As a result, the CMRA does not work like a standard ascending auction because even at high clock prices bidders can reveal demand at low prices using the additional bids. The auction ends when there exists a revenue-maximizing allocation in which *exactly one bid from every bidder* is accepted.

There are two key differences between the CMRA and the CCA, arguably the most frequently used combinatorial auction format in spectrum sales: the CMRA has no supplementary sealed-bid round (while the CCA does) and the CMRA allows for non-linearly priced additional bids in the clock stage (while the CCA does not). In the presence of complements, some form of non-linear pricing might be necessary to clear the market (Bikhchandani and Ostroy, 2002). To this end, the CCA typically uses Vickrey–Clarke–Groves (VCG)-like pricing in the supplementary round. By contrast, the CMRA allows bidders to submit non-linear (pay-as-bid-priced) additional bids in the clock phase. Hence, the market can clear in the clock phase, making the supplementary phase unnecessary.[1] Another difference is that the (clock phase of the) CCA ends when there is no excess demand for any good whereas the CMRA ends when the auctioneer can accept a bid from each bidder in a revenue-maximizing allocation.

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[1]The combinatorial auction due to Porter et al. (2003) also features no supplementary round.

There are several reasons why one might prefer to use an auction format with some of the features of the CMRA in practice. First, the CCA's supplementary round and pricing rule create uncertainty about what the bidders will win and what they will pay. Avoiding the supplementary round can therefore help bidders discover prices more gradually and deal more easily with their budget constraints (Janssen et al., 2017). Second, submitting mutually exclusive bids in the supplementary round (in order to achieve the most preferred auction outcome) can be taxing for bidders. Indeed, bidders might prefer the ability to revise the packages they bid on as clock prices change. Since the CMRA elicits demand information gradually, one might hope to overcome the "missing bids" problem that has been observed in CCAs (Bichler et al., 2013).

While a number of theoretical papers have explored the CCA, there has not yet been an analysis of the CMRA. This paper attempts to fill this gap.

In our baseline model, there are two bidders and a single divisible good. We allow a cap, which is a typical feature in spectrum auctions; no single bidder is allowed to win the entire supply. Bidders' marginal utilities for the good can be decreasing (capturing substitutes) or non-decreasing (capturing independent goods or complements). When we look at strategic bidding, we focus on equilibria in proxy strategies (Ausubel and Milgrom, 2002).

As a warm-up, we examine the effects of truthful (non-strategic) bidding. We identify two variants of truthful bidding in the CMRA. First, the headline demands can be truthful while no additional bids are placed. We call this strategy *clock-truthful* as it mimics truthful bidding in a clock auction. The second variant of truthful bidding features truthful headline demands and additional bids. Specifically, under this so-called *CMRA-truthful* strategy, bidders submit additional bids that give them the same surplus as the headline demand (so bidders are indifferent between winning with their headline demand or any additional bid). As bids can only be non-negative, bidders do not immediately bid on all packages; the range of packages that receives non-negative bids increases as the clock price goes up. The CMRA-truthful strategy was explicitly proposed by the auction designers (DotEcon, 2016).

The first insight is that in the presence of truthful additional bids the auction ends at a lower clock price than in their absence. Under CMRA-truthful bidding, bidders express true marginal values on a larger range of quantities at lower clock prices. It is then natural that the auction ends at a lower clock price and with lower revenue. Indeed, we show that both properties hold irrespective of bidders having increasing or decreasing marginal values (Propositions 1 and 2), except for an ambiguous revenue ranking in the non-decreasing marginal values case with asymmetric caps.

The second insight is that truthful additional bids can restore efficiency. Consider clock-truthful bidding when bidders have non-decreasing marginal values. Bidders demand the maximum capped quantity until they are indifferent between winning it and

winning nothing; they then drop demand to zero. Hence, during the auction they do not express any marginal values between zero and the maximum allowed quantity; the auction ends with excess supply. However, under CMRA-truthful bidding bidders use the additional bids to report their marginal values for quantities they never bid on in the clock auction, which leads to the efficient allocation. Propositions 1 and 2 show that CMRA-truthful bidding leads to the efficient allocation irrespective of increasing or decreasing marginal values.

Our main results concern strategic bidding in the CMRA. When marginal values are non-decreasing, CMRA-truthful bidding is an ex-post equilibrium when the bidders' caps are exactly symmetric. (This continues to be the case with any number of bidders.) With (arbitrarily small) asymmetries in the bidders' caps, however, CMRA-truthful bidding is no longer an equilibrium (Theorem 1). CMRA-truthful bidding is not an equilibrium with decreasing marginal values due to a free-riding and threshold problem (Proposition 3). Moreover, clock-truthful bidding is never an equilibrium: with decreasing marginal values, bidders have demand reduction incentives as in other multi-unit auctions (Ausubel et al., 2014) and with non-decreasing marginal values the weaker bidder has an incentive to close the auction early by bidding on the units left over by the stronger bidder.

Next, we identify a simple strategy that forms an ex-post equilibrium (Theorem 2). With symmetric caps, the (what we call *constant*) strategy prescribes a constant headline demand for the maximum (capped) quantity and a single additional bid of zero (i.e., the reserve price) for the residual supply at a relatively high clock price. The strategy thereby resembles clock-truthful bidding under non-decreasing marginal values except for the single additional bid. In this equilibrium and with decreasing marginal values, the bidders expand demand so that the stronger bidder wins a larger quantity than what she would have won in the efficient allocation. With non-decreasing marginal values, the equilibrium outcome is as under CMRA-truthful bidding and also as in a VCG auction under truthful bidding. With asymmetric caps, both bidders' headline demand is the second highest cap and the ex-post equilibrium exists as long as the utility for quantities between the two caps is not too different.

A final and crucial theoretical insight for practical auction design is that bidders can use additional bids to collude. Suppose each bidder plays the constant strategy, but also places an additional bid of half the initial clock price for half of the available quantity in the initial clock round. As both bidders winning half of the supply is (weakly) revenue-maximizing, the auction ends immediately by assigning half of the supply to each bidder at the initial clock price. We show that this strategy profile forms an equilibrium if bidders are sufficiently symmetric (Theorem 3). Surprisingly, this collusive demand reduction strategy carries *no risk*. If the other bidder does not work out the collusive bidding strategy and the auction does not end in the first round, the bidder is unconstrained in her bids in later rounds. In any case, the weaker bidder loses nothing by placing a zero

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additional bid on half of the supply.

We then argue that the driving force behind the collusion incentives is the activity rule used in the CMRA. Activity rules typically prevent bidders from underreporting their demand early in the auction (Ausubel and Baranov, 2020; Milgrom, 2000). However, the activity rule currently used in the CMRA only transforms headline demands into constraints on the bid function. We propose an alternative activity rule that makes the risk-free demand reduction strategy infeasible: Our alternative activity rule treats additional bids and headline demands identically by transforming both into constraints on bidding. The alternative activity rule keeps the other equilibria we discussed intact; however, it might force the bidders to revise the bids on more packages throughout the auction.

Finally, we connect our theoretical predictions to outcomes of CMRAs in practice. We look at the outcomes of three Danish spectrum auctions held between 2016 and 2021. As the DEA only publishes data on allocations and total bidder payments, we try to reverse-engineer whether headline demands or additional bids were winning in the auctions. We then use these insights to speculate whether the possible bidding dynamics were consistent with our theoretical predictions. It appears that in the 2016 auction only headline demands won, which suggests that bidders behaved as if they were in a clock auction. In the 2019 and 2021 auctions, some additional bids were winning, suggesting that bidders were using much richer strategies in later auctions. However, we do not find any evidence that bidders use the risk-free collusion strategy that we outline. We also explain what our results mean for patterns of bidding dynamics and how our theoretical predictions could be tested on actual CMRA bidding data.

The literature on combinatorial auctions is vast. Combinatorial auctions were introduced in the context of selling airport take-off and landing slots by Rassenti et al. (1982). Following the success of the SMRA for early US spectrum auctions in 1990s (Milgrom, 2000), license complementarities in subsequent sales were substantial enough to warrant new auction formats. As a result, a number of combinatorial auction designs were proposed (Parkes and Ungar, 2002; Porter et al., 2003).[2]

A major breakthrough for practical auction design was the development of the CCA (Ausubel et al., 2006; Maldoom, 2007) which quickly became one of the dominant auction formats for spectrum sales and beyond. The CCA was theoretically analyzed by Levin and Skrzypacz (2016), Janssen and Karamychev (2016) and Janssen and Kasberger (2019). We use many modelling features from the elegant analysis of Levin and Skrzypacz (2016) as well as their focus on proxy strategies. Unlike many other papers, we consider both increasing and decreasing marginal values.

On the theoretical side, many papers have proposed innovative dynamic auction for-

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[2]Palacios-Huerta et al. (2021) offer an excellent recent overview of the use of combinatorial auctions, including CMRAs, in practice.

mats for homogeneous goods (Ausubel, 2004), heterogeneous substitutes (Ausubel, 2006), heterogeneous complements (Sun and Yang, 2014; Baranov et al., 2017) as well as general valuations (Mishra and Parkes, 2007) in which truthful bidding is an ex-post equilibrium and final payments that coincide with VCG payments.[3] However, most of these formats are not used in practice. What makes the CMRA interesting to study is that it (i) has been used in practice; (ii) has an unusual format due the additional bids; (iii) admits a variety of ex-post equilibria, but (iv) has a truthful equilibrium only in certain cases.

A large literature has studied demand reduction in multi-unit auctions (e.g., Brusco and Lopomo (2002); Grimm et al. (2003); Ausubel et al. (2014)). The main difference is that in our case the demand reduction strategy is risk-free while in other auction formats it is not. We also observe demand *expansion* in equilibrium, which, to the best of our knowledge, has not been noted in auctions with pay-as-bid pricing.

We proceed as follows. The next section illustrates the dynamics of the clock auction and the CMRA under different bidding strategies. Section 3 presents the formal model and the auction rules. In Section 4, we analyze non-strategic truthful bidding and then turn to strategic bidding in Sections 5. Section 6 describes the risk-free collusive strategy and proposes an alternative activity rule to deal with it. Section 7 explains how to use bidding data to test which bidding strategies might have been played. We describe how the CMRA performed in the Danish spectrum auctions in Section 8. Section 9 concludes. All omitted proofs are in Appendix A. The extension to the case of any number of bidders is Appendix B. Further details about the Danish auctions are in the online appendix.

2 Illustrative Example: Clock Auction vs. CMRA

To make the CMRA more accessible, we present a simple example that demonstrates its rules and workings. As the CMRA builds on the clock auction, we first contrast the CMRA with the simplest possible clock auction to shed some light on the role of the CMRA's additional bids. As we explore the example further, we illustrate some of our main findings.

There are four identical lots for sale and two bidders. Bidders are symmetric: The valuation of each lot is $30 and the valuations are additive for each bidder. There is a symmetric cap of three lots for either bidder.

In a (standard) clock auction, the price starts at $0 and increases as long as there is excess demand. Under truthful bidding, each bidder demands three lots as long as the clock price is below $30. At $30 the clock stops because both bidders drop their demands to 0. The bidders have expressed demand for three units which can be allocated to either bidder, therefore there is excess supply of one lot. Hence, the final allocation is inefficient.

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[3]Sun and Yang (2009) and Teytelboym (2014) proposed dynamic auctions for selling restricted classes of substitutes and complements without offering a strategic analysis.

The auction revenue is $90.[4]

Next, consider how the CMRA would run. Bidders can mimic their strategies in the clock auction by only submitting a headline demand (and no additional bids) at each clock price. If the headline demands are truthful—this is the *clock-truthful* setting—and the price is below $30, then the revenue-maximizing allocation involves only *one* bidder; therefore, the price continues to increase. At $30, bidders drop their headline demand to zero lots. Now the revenue-maximizing allocation can involve a bid from both bidders: we can allocate three units to one bidder and no units to the other bidder. Therefore, the outcome is the same as in the clock auction. It is easy to see why clock-truthful bidding does not constitute an equilibrium: Bidders do not get any surplus from winning three lots, so a better strategy is to submit a headline demand of one unit in the initial round. This would end the auction immediately with both bidders having a positive payoff.

Now consider the presence of additional bids in the CMRA (headline demands continue to be three units at prices below $30). Figure 1 illustrates the *CMRA-truthful* strategy in our example. At any clock price up to $10, the bidder only submits a headline demand of three units. At the clock price of $10 the bidder is indifferent between winning three lots at linear prices (with a surplus $3 \times \$20 = \$60$) and winning two lots at a price of zero. Hence, the bidder submits an additional bid for a total of $0 for two lots (note that the price in this additional bid is below the clock price). Indeed, if both bidders submit such bids, there exists an allocation of all four lots to both bidders such that exactly one bid is accepted from each bidder, but it is not revenue maximizing (i.e., it raises $0 compared to the allocation of three units to one bidder which raises $30). Hence the auction continues. As the clock price increases, the bidder changes the additional bid for two lots. For example, at a clock price of $12, the bidder submits an additional bid of $6 for two lots. At a clock price of $20, the bidders place an additional bid of $30 for two lots and add an additional bid of $0 for one lot; the surplus from each bid is $30. The auction now ends because there are now revenue-maximizing allocations in which exactly one bid of each bidder is accepted: either the bidders' additional bids on two lots are accepted or one bidder's headline demand of three lots and one bidder's additional bid for one lot are accepted. In any case, the final allocation is efficient but revenue is lower ($60 vs. $90) than in the clock auction. We show that, perhaps surprisingly, CMRA-truthful strategies constitute an ex-post equilibrium in proxy strategies whenever bidders' marginal values are constant (as in this example) or increasing and the caps are symmetric. However, the CMRA-truthful equilibrium does not survive *any* asymmetry in bidders' caps.

There turns out to be another equilibrium in which the bidders can risk-freely collude.

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[4]Depending on the design of the clock auction, it is possible that when bidders drop their demand to zero simultaneously, the auctioneer allocates nothing to the bidders. In this case, imagine that the valuation of one of the bidders is $30+\epsilon$. Note that there are also installations of the clock auction that would allow market clearing.
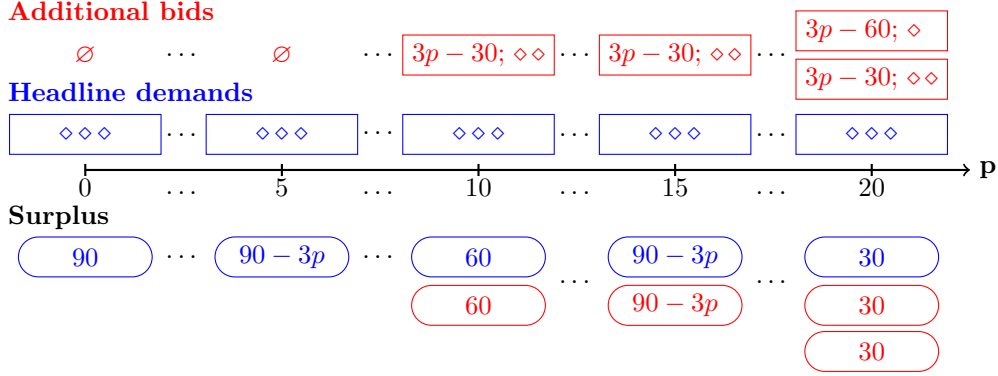
In the first round of the auction, the bidders follow the CMRA-truthful strategy, but also add an additional bid for two lots at a price of \$0. The auction ends immediately and both bidders win two units each at a price of zero. If the auction does not end in the first round, the bidders submit CMRA-truthful bids in all subsequent rounds and the auction will proceed as in case of CMRA-truthful equilibrium. Note that this first-round demand reduction carries no risk for the bidders in the sense that bidders do not have to give up "eligibility" to (potentially) end the auction early. In other words, they do not have to lower their headline demands to implement a collusive outcome. The clock auction, on the other hand, allows demand reduction only in headline demands; hence, lowering demand to two lots in early rounds does not allow the bidder to win more units later. In Section 6, we propose an alternative activity rule that prevents bidders from following this collusive strategy.

The model used in the paper is different from the illustrative example in three ways. First, we assume for tractability that the good is divisible. Second, we do not assume that marginal values are constant. Instead, we also allow for non-decreasing and increasing marginal values. Third, we allow for bidders to have asymmetric caps and incomplete information about each others' types.

3 The Combinatorial Multi-Round Ascending Auction

There are two bidders (denoted by $i \in \{1, 2\}$) and one unit of a perfectly divisible good whose quantity is denoted by $x \in [0, 1]$. (We deal with the case of any number of bidders in Appendix B.) In terms of the auction rules, we map the rules of the 2016 Danish

spectrum auction for the 1800 MHz band to our setting.

Clock and headline demands. The core of the CMRA is a clock auction: There is a price clock $p \in \mathbb{R}_+$ that ticks upwards, and bidders report their demands at each clock price p. The demanded quantity is called the *headline demand*. Bidder i's headline demand at clock price p is denoted by $h_i(p) \in [0, 1]$.

Additional bids. An innovation of the CMRA is that bidders can submit *additional bids* in each clock round. Formally, the additional bid for quantity x when the clock price is p is the price (willingness-to-pay) $A_i(x; p) \in \mathbb{R}_+ \cup \{-\infty\}$. In contrast to the headline demands, the additional bids do not have to be linear in clock prices. Instead, they must be below clock prices, i.e., $A_i(x; p) \leq px$ and must satisfy the activity rule described below. For quantities that do not receive any additional bids, we define the additional bids to be negative, i.e., $A_i(x; p) = -\infty$ if bidder i has not submitted an additional bid on quantity x. The auction rules specify that bids on the empty package must be non-positive.

At each clock price, the headline demand and the additional bids create a bid function $B_i: [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+ \cup \{-\infty\}$ that maps quantities and the clock price to bids. The bid function $B_i(\cdot; p)$ is then the collection of highest bids expressed in the course of the auction by bidder i at clock price p. For the headline demand the bid function simply gives us $B_i(h_i(p); p) = p \cdot h_i(p)$ while for the additional bids the bid function is defined as $B_i(x; p) = \sup_{\tilde{p} \leq p} A_i(x; \tilde{p})$.

Closing rule. The auction ends if there is a feasible revenue-maximizing allocation in which a bid by every bidder is accepted. Formally, the auction ends at clock price p if there is a non-negative allocation $(x_1, x_2) \in [0, 1]^2$, where (x_1, x_2) is such that

1. $(x_1, x_2) \in \arg \max_{(\tilde{x}_1, \tilde{x}_2) \in [0,1]^2, \, \tilde{x}_1 + \tilde{x}_2 \leq 1} B_1(\tilde{x}_1; p) + B_2(\tilde{x}_2; p)$, and

2. $B_i(x_i; p) \geq 0$ for $i \in \{1, 2\}$.

The second requirement guarantees that bidders win nothing only if they bid on zero quantity because the bid on the empty package must be zero by the auction rules above. If there is no such allocation, the auction continues by raising the clock price p.

It is worth comparing the closing rule to those of the clock auction and the CCA. The clock auction ends as soon as there is no excess demand. A bidder wins nothing only if she drops her demand to zero. The final allocation need not maximize revenue as it does not take past bids into account (unlike the CMRA). The clock phase in the CCA ends when there is no excess demand. Depending on the specific activity rules and the final clock round, it is possible that bidders do not win anything. The CCA selects the

final allocation as in closing rule condition (1) above (i.e., ignoring condition (2)), but the bidders' payments are weakly lower than B_i due to VCG-pricing.

Activity rule. The activity rule currently used in the CMRA translates the headline demands into constraints on the additional bids. Let $p < p'$ and $h_i(p) > h_i(p')$. Suppose p was the highest price at which $h_i(p)$ was demanded and that for all prices $\tilde{p} > p$ bidder i demanded $h_i(p')$. For $x \in (h_i(p'), h_i(p))$ the "relative cap" activity rule constrains $B_i(x; \tilde{p})$ from above by[5]

$$B_i(x; \tilde{p}) \leq B_i(h_i(p'); \tilde{p}) + p'(x - h_i(p')).$$

Strategies. A *proxy strategy* $(h_i, (A_i(\cdot; p))_{p \geq 0})$ for bidder i consists of a headline demand h_i and a collection $(A_i(\cdot; p))_{p \geq 0}$ of additional bids indexed by clock prices p. Similar to Levin and Skrzypacz (2016), we study equilibria in proxy strategies to focus on outcomes that are specific to the CMRA. Levin and Skrzypacz (2016) point out that dynamic auctions typically feature other equilibria in which certain actions trigger reactions of the other bidders. The restriction to proxy strategies avoids the specification of (updated) beliefs about the other bidder's type, which does not play a major role in our analysis due to our focus on ex post equilibria.[6]

Information. In each round, bidders learn (i) the current clock prices, (ii) whether their previous headline demand appeared in a revenue-maximizing allocation, (iii) whether they appeared in a revenue-maximizing allocation. Due to our focus on proxy strategies, (ii) and (iii) will play no role to our analysis.[7]

Payoffs. Bidder i has a quasilinear utility function $U_i(x) - px$, where $U_i(x)$ is the value of winning quantity x. Let $u_i(x)$ denote the marginal value, so $U_i(x) = \int_0^x u_i(y)dy$. Marginal values are strictly positive. There is a *cap* $\lambda_i \in (1/2, 1)$ such that bidder i can win at most λ_i.[8] Without loss of generality, let $\lambda_1 \geq \lambda_2$. Spectrum caps were present in all real-world implementations of the CMRA. Note that some auctions feature symmetric caps while other auctions restrict bidders asymmetrically. Hence, both cases are of interest.

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[5]If headline demand is x and lowered continuously at clock price p, then the slope of the bids at x can be at most p.

[6]Moreover, proxy strategies only specify "on-path" behavior; as the proxy strategy rules out certain bidding histories, the strategy does not specify behavior after such histories. Specifically, there are sequences of additional bids that put non-trivial constraints on the bidding rules. Specifying behavior in such cases is notationally complex. We avoid these issues by considering proxy strategies.

[7]Note that in the 2021 Danish spectrum auction bidders learned (i)–(iii), while in the 2016 and 2019 auctions they only learned (i).

[8]An alternative interpretation is that λ_i is bidder i's capacity (Ausubel et al, 2014) or satiation point where marginal values are at or below the reserve price. In this case, we would define $U_i(x)$ to be strictly increasing on $[0, \lambda_i]$ and flat on $(\lambda_i, 1]$, that is, $U_i(x) = U_i(\lambda_i)$ for $x \in [\lambda_i, 1]$.

The value functions are twice continuously differentiable and parameterized by private types θ_i, i.e., $U_i(x) = U(x; \theta_i)$, where $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$ and $\underline{\theta}_i, \overline{\theta}_i \in \mathbb{R}_+$. We say that bidder i is *stronger* than bidder j if $\theta_i \geq \theta_j$. Let values be strictly increasing in θ_i, i.e., $\partial U(x; \theta_i)/\partial \theta_i > 0$ for all $x \in [0, \lambda_i]$. Let the marginal values be strictly increasing in θ_i, that is, $\partial U/\partial x \partial \theta_i > 0$. We assume that there are at least three types, but do not otherwise specify the joint type distribution due to our focus on ex-post equilibria.

Our analysis distinguishes decreasing marginal values (capturing substitutes) and non-decreasing marginal values (capturing independent goods and complements). If bidders have decreasing marginal values on $[0, \lambda_i]$, then we also assume that the efficient allocation is interior, i.e., $1 - \lambda_j < x_i^\star < \lambda_i$.[9] The following assumption contains a sufficient condition.

Assumption 1 (**Decreasing Marginal Values**). Bidders have *decreasing marginal values* if $u_i'(x) < 0$ for $x \in [0, \lambda_i]$ and $u_i(\lambda_i) < u_j(1 - \lambda_i)$ for $i, j = 1, 2$ and all feasible type profiles.

When bidders have decreasing marginal values, the efficient allocation $x^\star$ solves $\max U_1(x_1) + U_2(x_2)$ subject to $x_1 + x_2 \leq 1$. The assumptions imply that $x_1^\star = 1 - x_2^\star$ and that $u_1(x_1^\star) = u_2(1 - x_1^\star)$.

We also analyze settings with non-decreasing marginal values.

Assumption 2 (**Non-decreasing marginal values**). Bidders have *non-decreasing marginal values* if $u_i'(x) \geq 0$ for $x \in [0, \lambda_i]$ for $i = 1, 2$.

The efficient allocation must lie on the boundary due to non-decreasing marginal values. Recall that we assume that $\lambda_1 \geq \lambda_2 > \frac{1}{2}$. If $\theta_1 \geq \theta_2$, then $(\lambda_1, 1 - \lambda_1)$ is the efficient allocation. To see this, observe that $U_1(\lambda_1) + U_2(1 - \lambda_1) > U_1(1 - \lambda_2) + U_2(\lambda_2)$ is equivalent to $\int_{1-\lambda_2}^{\lambda_1} u_1(x)dx > \int_{1-\lambda_1}^{\lambda_2} u_2(x)dx$. Both integrals are over the same mass but the left-hand side integrates over higher quantities. As the marginal values are increasing in θ, the inequality holds. In the case of symmetric caps, a similar argument shows that it is efficient that the stronger bidder wins λ. Now, if $\theta_2 > \theta_1$, then in principle both allocations $(\lambda_1, 1 - \lambda_1)$ and $(1 - \lambda_2, \lambda_2)$ can be efficient. Intuitively, $(\lambda_1, 1 - \lambda_1)$ can be efficient if $U_1(\lambda_1)$ is much higher than $U_2(\lambda_2)$ due to the asymmetric caps and increasing marginal values while there may be not much difference between U_1 and U_2 for $x \leq \lambda_2$.

4 Outcomes under non-strategic truthful bidding

In this section, we investigate outcomes under non-strategic truthful bidding. Our motivation is three-fold. First, truthful strategies allow us to introduce the workings of the CMRA without the need to keep track of incentives. Second, truthful strategies provide

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[9]Corner solutions in which one bidder is much stronger than another merely add to casework without affecting the economic content of our results.

a natural benchmark in auctions with many bidders. Lastly, these strategies set the stage for the strategic incentives studied in the next section.

We compare the simple clock auction to the CMRA to isolate the effect of the additional bids. As the CMRA is built on a clock auction, any clock auction behavior is also feasible in the CMRA; bidders only have to ignore the possibility of additional bids.

In the clock auction, the meaning of truthful bidding is unambiguous. Bidders can demand only a single quantity at a time, and this quantity is utility-maximizing if it is truthful. Similarly, the headline demand $h_i(p)$ in the CMRA is *truthful* at current price p if $h_i(p) \in \arg\max_x U_i(x) - px$. We further say that a bidding strategy in the CMRA is *clock-truthful* if the bidder only submits truthful headline demands at all clock prices and submits no additional bids.

The CMRA's additional bids allow richer behavior that may be considered truthful. Truthful headline demand at all prices elicits a demand curve that reflects the bidder's marginal values. This motivates us to interpret truthful bidding in the CMRA as expressing true marginal values for all shares for which this is possible at a given clock price. The following "truthful" bidding strategy was described by DotEcon:

> *The proposed auction design allows bidders to follow alternative bid strategies that give them more control over their risks. For instance, a bidder may submit many bids at all times with a view to maximise surplus. Making a headline bid for the surplus-maximising package and then for all other packages with a bid amount equal to the value of the package minus the surplus on the headline bid would achieve this.* (DotEcon, 2016)

In other words, DotEcon suggests that bidders place an additional bid for x such that the bidder is indifferent between winning x for a payment of $A_i(x; p)$ and the headline demand at price p. Intuitively, the bidder would then bid true marginal values for all shares for which this is possible with non-negative bids. Formally, we say that a bidding strategy is *CMRA-truthful* if the headline demands are truthful and

$$A_i(x; p) = \begin{cases} U_i(x) - V_i(p) & \text{for } x \in [U_i^{-1}(V_i(p)), \lambda_i] \\ -\infty & \text{otherwise,} \end{cases} \tag{1}$$

where $V_i(p) = \max_x U_i(x) - px$. The additional bids are generically non-linear in clock price p.

The CMRA-truthful bids have the following properties. For $p = 0$, the truthful demand is λ_i so that $V_i(0) = U_i(\lambda_i)$ and $U_i^{-1}(U_i(\lambda_i)) = \lambda_i$; only λ_i receives a non-negative bid at $p = 0$. The envelope theorem implies that $V_i(p)$ decreases in p, implying that the range of quantities that receive non-negative bids increases in p. Moreover, the bid on a given quantity increases in the clock price. Note that as p tends to infinity,

$V_i(p)$ converges to 0. Thus, for sufficiently high prices the bidder bids on all quantities in $[0, \lambda_i]$ and the additional bids are the true valuation levels. Hence, the domain of the non-negative additional bids is well defined. For sufficiently low p, the bidders shade their bids by bidding less than value without distorting marginal values for quantities for which a positive bid is made.[10]

4.1 Decreasing marginal values

We now analyze auction outcomes when bidders have decreasing marginal values.

Proposition 1. *Suppose that marginal values are decreasing.*

(i) *Clock-truthful bidding in the CMRA leads to the efficient allocation. The clock ends at clock price $p^\star = u_1(x_1^\star) = u_2(x_2^\star)$.*

(ii) *CMRA-truthful bidding leads to the efficient allocation and ends the auction at a clock price lower than $p^\star$.*

(iii) *Ex-post revenue under CMRA-truthful bidding is lower than under clock-truthful bidding.*

With decreasing marginal values, truthful bidding in the clock auction and clock-truthful bidding in the CMRA both lead to the efficient allocation. The revenue is $p^\star$. CMRA-truthful bidding also leads to the efficient allocation but ends the auction at a lower price and at lower revenue. The reason why the CMRA ends earlier than the clock auction with the efficient allocation is that bidders express true marginal values for more quantities at lower clock prices.

Figure 2 illustrates Proposition 1 in the symmetric-caps case. Figure 2a shows the headline demands and additional bids at various clock prices. Bidder 1 is stronger and her efficient share is $x_1^\star > \frac{1}{2}$ while bidder 2's efficient share is $x_2^\star < \frac{1}{2}$. Solid lines are headline demands $h_i(p)$ and dashed lines are additional bids $A_i(x; p)$ as in Eq. (1). Figure 2b depicts the respective revenue from feasible allocations. The solid line $B_1(x; p) + B_2(1 - x; p)$ shows revenue for allocations in which a bid of each bidder is accepted since this is required by the CMRA closing rule (recall that bids are $-\infty$ for shares that bidders do not bid on). The dashed line is $\max\{B_1(x; p), B_2(1 - x; p)\}$ for allocations that do not receive non-negative bids from both bidders: this is revenue that can be obtained by accepting only one bidder's bid.

Let us consider how the bids and allocations change as the clock price increases. As a benchmark, consider a simple clock auction (or a CMRA with clock-truthful bidding).

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[10]The CMRA-truthful strategy can be seen as a family of "truncation reports" as defined by Day and Milgrom (2008).

We simply increase the clock price and follow the headline demands in solid lines in Figure 2a. The auction ends at clock price $p^\star$ with market clearing.

Under CMRA-truthful bidding, both bidders submit headline demands and additional bids. When the clock price p is low, only quantities close to λ receive additional bids.

- *Clock price p_1.* At this clock price, each bidder's headline demand is λ, yielding surplus $U_i(\lambda) - \lambda p_1$. Recall that the additional bids are given by Eq. (1). At p_1, the additional bids range from λp_1 (for a quantity λ) to zero (for a smaller quantity that keeps the bidder indifferent). At p_1, there is no feasible allocation that receives bids from both bidders (the dashed lines do not intersect in Figure 2b). The auction continues as it is not possible to accept a bid by each bidder in the revenue-maximizing allocation.

- *Clock price p_2.* This is the lowest price at which both bidders bid on their respective efficient quantities. Bidder 1 submits a strictly positive additional bid on $x_1^\star$ at clock price p_2, while bidder 2 submits an additional bid of 0 on $x_2^\star$. From now on the efficient allocation can in principle be allocated as it receives bids from both bidders. Figure 2b reveals, however, that the efficient allocation is not revenue-maximizing. Bidder 1's headline demand is still λ, and allocating λ to bidder 1 raises a revenue of λp_2. Observe that bidder 2 bids less than λp_2 on λ because marginal values are decreasing and because λ is not the headline demand. As bidder 2 does not bid on $1 - \lambda$ at clock price p_2, the revenue-maximizing allocation features only bids by one bidder and the auction continues.

- *Clock price p_3.* Both bidders have now raised their additional bids on their respective efficient share. As we can see in Figure 2b, the efficient allocation $x^\star$ locally maximizes revenue. However, $x^\star$ does not yield a global revenue maximum as bidder 1's bid on λ leads to a higher revenue of λp_3.

- *Clock price $\tilde{p}^\star$.* The additional bids are now sufficiently high so that the efficient allocation $x^\star$ is revenue-maximizing. The auction ends at $\tilde{p}^\star$ at which

$$B_1(x_1^\star; \tilde{p}^\star) + B_2(x_2^\star; \tilde{p}^\star) = \max_i B_i(\lambda; \tilde{p}^\star). \tag{2}$$

Note that at price $\tilde{p}^\star$ bidder 2 does not yet bid on $1 - \lambda$, so $(\lambda, 1 - \lambda)$ is not a feasible allocation. Revenue is lower than $\tilde{p}^\star$ and lower than $p^\star$ (Fig. 2b).

4.2 Non-decreasing marginal values

We now consider with non-decreasing marginal values a setting which has received less attention in the literature than decreasing marginal values.
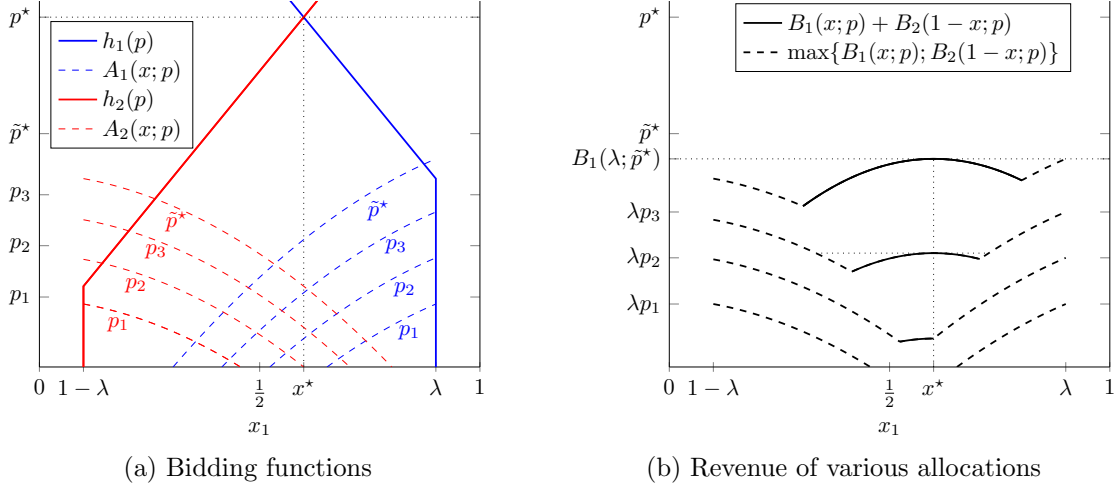
| (a) Bidding functions | (b) Revenue of various allocations |

Figure 2: CMRA-truthful bidding with decreasing marginal values.

Note: We depict $\max\{B_1(x;p), B_2(1-x;p)\}$ only if the allocation $(x, 1-x)$ has not received bids from both bidders.

Proposition 2. *Suppose that marginal values are non-decreasing.*

(i) *Clock-truthful bidding in the CMRA leads to excess supply. The clock ends at clock price $p = \min_i U_i(\lambda_i)/\lambda_i$. The final auction allocation is inefficient.*

(ii) *CMRA-truthful bidding clears the market and leads to the efficient allocation.*

(iii) *If $\lambda_1 = \lambda_2$, then ex-post revenue under CMRA-truthful bidding is lower than under clock-truthful bidding; if $\lambda_1 > \lambda_2$, then the ex-post revenue comparison is ambiguous.*

Proposition 2 reveals the potentially efficiency-restoring effect of additional bids in the CMRA. Truthful additional bids lead to market-clearing and an efficient allocation while their absence leads to inefficiencies and excess supply. After providing intuition for the result, we provide an example that shows that the revenue-ranking can be reversed with asymmetric caps.

Figure 3 illustrates Proposition 2 for symmetric caps. As before, Figure 3a shows the headline demands and the additional bids, while Figure 3b shows revenue under different allocations.

Once again, let us first consider the outcome of a clock auction or of clock-truthful bidding in the CMRA. Figure 3a shows that, due to increasing marginal values, bidder i's clock-truthful headline demand is λ for $p \leq U_i(\lambda)/\lambda$ and 0 for higher prices. Hence, the auction ends at clock price $p = \min_i U_i(\lambda)/\lambda = U_2(\lambda)/\lambda$. As bidder 2 drops demand to 0 at price $U_2(\lambda)/\lambda$, the auction ends with excess supply of $1 - \lambda$. Bidder 1 wins quantity λ and the revenue is $U_2(\lambda)$.

Under CMRA-truthful bidding, bidders' headline demands are as under clock-truthful bidding but they also submit additional bids. For low clock prices, bidders submit few
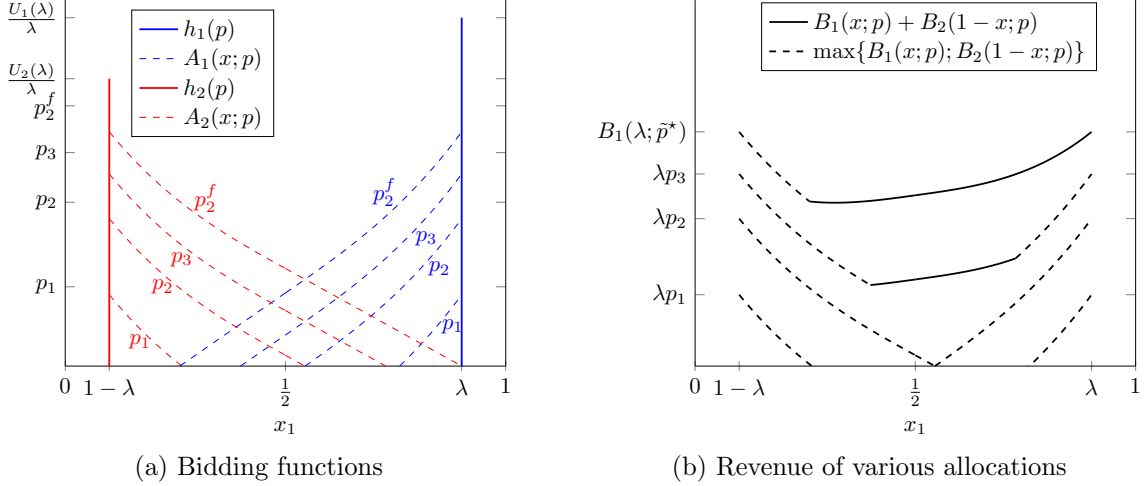
(a) Bidding functions

(b) Revenue of various allocations

Figure 3: CMRA-truthful bidding with non-decreasing marginal values

Note: We depict $\max\{B_1(x;p), B_2(1-x;p)\}$ only if the allocation $(x, 1-x)$ has not received bids from both bidders.

additional bids, but as the clock price rises, bidders increase their additional bids both on the intensive and extensive margins.

- *Clock price p_1.* There is no feasible allocation that receives non-negative bids from both bidders. Hence, the auction continues.

- *Clock price p_2.* There are now feasible allocations that receive non-negative bids from both bidders. These allocations are not revenue-maximizing as $B_i(\lambda; p) = \lambda p$ yields higher revenue.

- *Clock price p_3.* Feasible allocations that receive non-negative bids from both bidders are still not revenue-maximizing. Bidder 1's marginal values (and additional bids) are higher and non-decreasing, so allocating more to bidder 1 increases revenue.

- *Clock price p_2^f.* At this price, the weaker bidder 2 places an additional bid of 0 on $1 - \lambda$. More generally, there is a *final price p_i^f* at which bidder i bids 0 on $1 - \lambda$ as this bidder is indifferent between winning λ for a payment $p_i^f \lambda$ and winning $1 - \lambda$ for free. The indifference condition $U_i(\lambda) - p_i^f \lambda = U_i(1 - \lambda)$ transforms to

$$p_i^f = \frac{U_i(\lambda_i) - U_i(1 - \lambda_j)}{\lambda_i}. \tag{3}$$

With bidder 2's additional bid, it is now possible to accept a bid by each bidder in the revenue-maximizing allocation $(\lambda, 1 - \lambda)$ (Figure 3b). Therefore, the CMRA ends in market-clearing. The revenue is λp_2^f, which is lower than $\lambda U_2(\lambda)/\lambda$.

The following numerical example illustrates that revenue can be higher under CMRA-truthful bidding and how the CMRA may end with asymmetric caps.

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Example 1 (Higher revenue under CMRA-truthful bidding). Let $\lambda_1 = \frac{7}{8}$, $\lambda_2 = \frac{6}{8}$, $U_1(\lambda_1) = 21$, $U_1(1 - \lambda_2) = 1$, $U_2(\lambda_2) = \frac{39}{2}$, and $U_2(1 - \lambda_1) = \frac{1}{2}$.[11] The allocation $(\lambda_1, 1 - \lambda_1)$ is efficient. Under clock-truthful bidding, the CMRA ends at clock price $\min_i U_i(\lambda_i)/\lambda_i = U_1(\lambda_1)/\lambda_1 = 24 < 26 = U_2(\lambda_2)/\lambda_2$. Consider CMRA-truthful bidding. Bidder 1 starts bidding on $1 - \lambda_2$ at clock price $p_1^f = \frac{160}{7}$. In contrast to the symmetric-caps case, the auction does not end at p_1^f because $(\lambda_1, 0)$ is the revenue-maximizing allocation at this clock price. The earliest clock price p at which $(1 - \lambda_2, \lambda_2)$ becomes revenue-maximizing solves

$$B_1(1 - \lambda_2; p) + B_2(\lambda_2; p) = U_1(1 - \lambda_2) - U_1(\lambda_1) + p\lambda_1 + p\lambda_2 = p\lambda_1.$$

The solution is $p = (U_1(\lambda_1) - U_1(1 - \lambda_2))/\lambda_2 = 80/3$. Note, however, that bidder 2 bids 0 on $1 - \lambda_1$ at clock price $p_2^f = 76/3$ ($p_1^f < p_2^f$). Hence, the auction ends at clock price p_2^f with the efficient allocation $(\lambda_1, 1 - \lambda_1)$. Revenue is $p_2^f \lambda_1 = \frac{133}{6}$, which is less than the revenue under clock-truthful bidding: $\lambda_2 U_1(\lambda_1)/\lambda_1 = 18$. ◄

5 Outcomes under strategic bidding

Having characterized the outcomes under non-strategic truthful bidding, we turn to the strategic incentives in the CMRA. We first show that neither form of truthful bidding is an equilibrium with decreasing marginal values. When marginal values are non-decreasing, CMRA-truthful (but not clock-truthful) bidding can be an equilibrium with symmetric caps. However, this equilibrium is fragile to small asymmetries in the caps. We then characterize a particularly simple but non-truthful equilibrium. In the next section, we illustrate how bidders can risklessly collude in the CMRA. The takeaway from these two sections is that the CMRA does not always provide incentives for truthful bidding and is instead vulnerable to many forms of strategic manipulation.

Throughout, we use ex-post equilibrium in proxy strategies as our solution concept. In such an equilibrium, bidders' strategies are proxy strategies that are best responses to the opponent's strategy and any type distribution. In particular, bidders would not choose an alternative proxy strategy if they knew the opponent's type.

5.1 No truthtelling with decreasing marginal values

We first examine whether the clock-truthful or CMRA-truthful strategies form equilibria when bidders have decreasing marginal values. Recall the CMRA's pricing rule: winning headline demands are priced at the clock price and winning additional bids are priced as bid. Such pricing features appear in clock auctions and (sealed-bid) discriminatory

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[11]It is straightforward to check that the utility functions are consistent with Assumption 2. The difference $\lambda_1 - (1 - \lambda_2) = \lambda_2 - (1 - \lambda_1) = \frac{5}{8}$ and $U_1(\lambda_1) - U_1(1 - \lambda_2) = 20 > 19 = U_2(\lambda_2) - U_2(1 - \lambda_1)$.

multi-unit auctions. A common property of equilibria in these auctions is that bidders have an incentive to "shade" their bids below their value (Ausubel et al., 2014). This intuition carries over to the CMRA when bidders have decreasing marginal values.

Remark 1. Clock-truthful bidding is not an equilibrium in the CMRA when bidders have decreasing marginal values just like truthful bidding is not an equilibrium in the clock auction. As bidders pay px for winning x at final clock price p, they have an incentive to report a marginally lower willingness-to-pay to end the auction at a lower price as this saves them money on units they win. Such (marginal) deviations are possible only if the other bidder uses a downward sloping demand curve as is natural with decreasing marginal values.

The following proposition states that CMRA-truthful bidding also does not form an equilibrium in the CMRA when bidders have decreasing marginal values.

Proposition 3. *Suppose that marginal values are decreasing. Then there is no ex-post equilibrium in proxy strategies in which bidder i bids CMRA-truthfully.*

The proof of the proposition can be illustrated by what happens at clock price p_3 in Figure 2. At this clock price, the bids are such that the efficient allocation is a local, but not global, revenue maximum. Bidders face a threshold problem[12] and face two kinds of deviation incentives captured by Eq. (2). First, bidders have an incentive to free-ride on their competitor by shading their bids on (what the bidders expect to be the) efficient shares even more, i.e., bidding lower than dictated by the truthful additional bids. In particular, bidders have a greater incentive to decrease the additional bids more on lower quantities in order to reduce their payments without affecting the allocation significantly. That is, in Eq. (2), bidder 1 prefers to keep her bid $B_1(x_1^\star; p)$ low and for bidder 2 to increase $B_2(x_2^\star; p)$. Second, bidders also have an incentive to shade their bids on high quantities. This allows stronger bidders to remove the threshold problem and close the auction early. In Eq. (2), a bidder might wish to decrease $B_i(\lambda_i; \tilde{p}^\star)$. As a result, CMRA-truthful bidding is not an equilibrium.

5.2 Fragile truthtelling with non-decreasing marginal values

We now turn to non-decreasing marginal values. Unsurprisingly, clock-truthful bidding is also not an equilibrium when the bidders have non-decreasing marginal values.

Remark 2. Clock-truthful bidding is not an equilibrium in the CMRA when bidders have non-decreasing marginal values. Under clock-truthful bidding, bidders first headline-demand λ_i and then drop demand to zero. The auction ends as soon as the first bidder

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[12]This is related to what happens in other combinatorial auctions where small bidders need to jointly outbid a larger competitor. In our case, the bidders may wish to coordinate additional bids on small quantities in order to outbid their bids on larger quantities.

does so. Hence, under clock-truthful bidding, weak bidders would expect to win nothing, therefore submitting a headline demand of $1 - \lambda_j$ at a clock price of 0 is a profitable deviation.

The weak bidders' profitable deviation described in Remark 2 might not be necessary under CMRA-truthful bidding because bidder i anyway submits an additional bid on $1 - \lambda_j$ and this additional bid can be as low as zero. Recall that bidder i bids 0 on $1 - \lambda_j$ at clock price p_i^f, which is the clock price at which bidder i is indifferent between winning λ_i for transfer $p_i^f \lambda_i$ and winning $1 - \lambda_j$ for free. In the symmetric-caps case, the bidder with the lower type makes the zero bid on $1 - \lambda$ first and this bid ends the auction. This proves to be sufficient for CMRA-truthful bidding to form an equilibrium. With asymmetric caps, the additional zero-bid does not always end the auction, which distorts the incentives away from truthful bidding.

Theorem 1. *Suppose that marginal values are non-decreasing.*

(i) *Let* $\lambda_1 = \lambda_2$. *The CMRA has an ex-post equilibrium in proxy strategies in which bidder i bids CMRA-truthfully.*

(ii) *Let* $\lambda_1 > \lambda_2$. *CMRA-truthful bidding does not form an ex-post equilibrium.*

Focusing on the symmetric-caps case, it may be surprising that CMRA-truthful bidding can ever be an (ex-post) equilibrium: Truthful bidding neither forms an equilibrium in the pure clock auction nor in the discriminatory auction. Yet, in the CMRA, a combination of these two auction formats, truthful bidding even forms an *ex-post* equilibrium.[13] However, the existence of this equilibrium is not robust to asymmetries in the bidders' caps. From a practical perspective, it is worth pointing out that some auctions feature symmetric caps. However, if λ_i denotes the bidder's capacity (see footnote 8), then the symmetry assumption may be more unlikely to hold.

Why is CMRA-truthful bidding an equilibrium when bidders have non-decreasing marginal values and symmetric caps? The truthful headline demand is λ as long as the clock price is not prohibitively high. Consider any deviating bid $D_i(x)$ for quantity x. Such a bid must be sufficiently high as it becomes winning only if it is part of a revenue-maximizing allocation. In particular, the deviating bid $D_i(x)$ must satisfy

$$D_i(x) + B_j(1 - x; p) \geq B_j(\lambda; p).$$

Plugging in the truthful bids of bidder j gives

$$D_i(x) \geq U_j(\lambda) - U_j(1 - x).$$

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[13]In Appendix B we show that CMRA-truthful bidding also forms an ex-post equilibrium when there are more than two bidders with non-decreasing marginal values and symmetric caps.

Observe that the right-hand side is precisely the VCG price in a VCG auction in which bidders bid truthfully. In such a VCG auction, the strong bidder i wins λ and pays $U_j(\lambda) - U_j(1 - \lambda)$, and the weak bidder j wins $1 - \lambda$ and pays $U_i(\lambda) - U_i(\lambda) = 0$. In the CMRA with truthful bidding and symmetric caps, bidders' payments are also exactly these VCG prices, which implies that bidders cannot profitably deviate.

Asymmetric caps change the auction outcome under CRMA-truthful bidding in a crucial way: The auction does not necessarily end with the first zero-bid on $1 - \lambda_j$. In particular, if the bidder with the higher cap (bidder 1) places the zero-bid on $1 - \lambda_2$ first, the auction does not end as bidder 1's headline demand is still revenue-maximizing but does not include bids from both players:

$$B_1(1 - \lambda_2; p_1^f) + B_2(\lambda_2; p_1^f) = p_1^f \lambda_2 < p_1^f \lambda_1 = B_1(\lambda_1; p_1^f).$$

Hence, the auction continues and drives up prices, which impacts the bidders' incentives. If bidder 1 eventually wins $1 - \lambda_2$, the payment is strictly positive and bidder 1 has an incentive to end the auction early by deviating to a headline demand of $1 - \lambda_2$ in the first round. On the other hand, if the type profile was such that $p_1^f < p_2^f$ and $(\lambda_1, 1 - \lambda_1)$ the final allocation, then the auction would end at clock price p_2^f under CMRA-truthful bidding. However, observe that bidder 1 then prefers winning $1 - \lambda_2$ for free over winning λ_1 for transfer $p_2^f \lambda_1$:

$$U_1(\lambda_1) - \lambda_1 p_2^f = U_1(\lambda_1) - \frac{\lambda_1}{\lambda_2}\left(U_2(\lambda_2) - U_2(1 - \lambda_1)\right) < U_1(1 - \lambda_2) \Leftrightarrow p_1^f < p_2^f.$$

Bidder 1 can win $1 - \lambda_2$ for free by submitting an additional bid of zero for $1 - \lambda_2$ at clock price 0. Note that bidder 1 would pay an *inflated* VCG price under CMRA-truthful bidding: $\lambda_1 p_2^f$ equals λ_1/λ_2 times the VCG price, where the coefficient $\lambda_1/\lambda_2 > 1$ appears due to the asymmetric caps. These arguments show that CMRA-truthful bidding can form an equilibrium only if $p_2^f \leq p_1^f$ for all type profiles; in this case, the auction certainly ends at p_2^f with bidder 2 making the zero-bid. Next, we show that this necessary condition is not sufficient.

CMRA-truthful bidding is not bidder 1's best response to bidder 2's CMRA-truthful strategy if $p_2^f < p_1^f$ for all type profiles. In particular, bidder 1 can lower the payment for λ_1 from $p_2^f \lambda_1$ to $p\lambda_1$, where $p < p_2^f$. In the proof, we also provide a lower bound on p to make the deviation feasible. The lower payment can be guaranteed by headline-demanding λ_1 until clock price p, not making any additional bids, and dropping demand to 0 at clock price p. This freezes the price bidder 1 has to pay for λ_1 to $p\lambda_1$. Due to the cap asymmetry and $p_2^f < p_1^f$, the auction does not end at p but continues until bidder 2 places the zero-bid on $1 - \lambda_1$ at clock price p_2^f. As bidder 1 can secure λ_1 for a lower payment, CMRA-truthful bidding does not form an equilibrium.

5.3 A non-truthtelling equilibrium for any marginal values

We now describe a bidding strategy which turns out to be an ex-post equilibrium irrespective of what the bidders' marginal values are. In the symmetric-caps case, the bidders submit a headline demand of λ for as long as it gives them non-negative surplus. There are no additional bids except for a single additional bid of 0 on $1 - \lambda$ at clock price p_i^f (at which bidder i is indifferent between winning λ and winning $1 - \lambda$ for free). As such, the strategy is "almost clock-truthful" when marginal values are non-decreasing. We refer to this strategy as the *constant bidding* strategy. Given that both players play this strategy, the stronger bidder i wins λ at a linear price of p_j^f and the weaker bidder j wins $1 - \lambda$ for free. With non-decreasing marginal values, the auction outcome is as under CMRA-truthful bidding, including the VCG prices. However, we know from Theorem 1 that bidders have incentives to deviate from this outcome when the caps are asymmetric.

For incentive compatibility in the asymmetric-caps case, the price at which the additional zero-bid is made must only be determined by bidders' types rather than also by their caps. This is achieved by bidders' bidding on $\lambda_2 = \min\{\lambda_1, \lambda_2\}$ as their headline demand. To this end, define $p^f(\theta) = (U(\lambda_2; \theta) - U(1 - \lambda_2; \theta))/\lambda_2$. To state the strategy in a unified way, let λ_2 stand for λ in the symmetric-caps case. In the *constant* strategy, bidder i's headline demand is

$$h_i(p) = \begin{cases} \lambda_2 & \text{for } p \leq \frac{U_i(\lambda_2)}{\lambda_2} \\ 0 & \text{otherwise;} \end{cases} \tag{4}$$

bidder i places a single additional bid of $A_i(1 - \lambda_2; p^f(\theta_i)) = 0$.

Theorem 2. *Suppose that both bidders either have decreasing or non-decreasing marginal values.*

(i) *Let $\lambda_1 = \lambda_2$. The CMRA has an ex-post equilibrium in proxy strategies in which bidder i plays the constant strategy.*

(ii) *Let $\lambda_1 > \lambda_2$. The CMRA has an ex-post equilibrium in proxy strategies in which bidder i plays the constant strategy if and only if $\overline{\theta}_1$ and $\underline{\theta}_2$ are such that*

$$U(\lambda_1; \overline{\theta}_1) \leq U(\lambda_2; \overline{\theta}_1) + U(1 - \lambda_2; \underline{\theta}_2). \tag{♠}$$

If the marginal values are decreasing, the final allocation is generically inefficient. If the marginal values are non-decreasing, (a) the outcome is inefficient if $(\lambda_1, 1 - \lambda_1)$ is efficient; (b) the outcome is efficient if $(1 - \lambda_2, \lambda_2)$ is efficient; and (c) the outcome is as under CMRA-truthful bidding if the caps are symmetric.

To confirm that the constant bidding strategy is a best response to itself, let $\theta_i \geq \theta_j$. Then bidder i knows that she wins λ_2 for a price of $p^f(\theta_j)$. She cannot win λ_2 at a lower

price. Winning λ_2 at $p^f(\theta_j)$ is better than winning $1 - \lambda_2$ for free by construction of p^f and $p^f(\theta_j) \leq p^f(\theta_i)$. She could win less than $1 - \lambda_2$, but this is not profitable. With asymmetric caps, bidder 1 may want to win λ_1 with the same deviation strategy as the one that proves that CMRA-truthful bidding does not form an equilibrium. Then the auction does not end at $p^f(\theta_2)$, but at bidder 2's drop-out price $p = U_2(\lambda_2)/\lambda_2$.[14] Bidder 1's deviating bid (and therefore payment) must be at least $U_2(\lambda_2)$ so that the deviation leads to a revenue-maximizing allocation. The deviation is then profitable only if

$$\underbrace{U_1(\lambda_1) - U_2(\lambda_2)}_{\text{deviation payoff}} \geq \underbrace{U_1(\lambda_2) - U_2(\lambda_2) + U_2(1 - \lambda_2)}_{\text{payoff from the constant strategy}}.$$

Condition (♠) implies that winning λ_1 is not profitable. Now, if $\theta_i \leq \theta_j$, then bidder i knows she wins $1 - \lambda_2$ for free. She cannot win $1 - \lambda_2$ at a lower price. She does not want to win more as this would be more expensive and therefore not profitable by construction of p^f.

Note that condition (♠) constrains only the highest type of bidder 1 and the lowest type of bidder 2; the equilibrium exists if the bidder with the higher cap cannot also have a much higher type than the low-cap bidder. Bidder 2 can have arbitrarily high types because bidder 2's cap eliminates the potential deviation of winning more than λ_2. Observe that condition (♠) can be satisfied for sufficiently similar caps for any given $\overline{\theta}_1$ and $\underline{\theta}_2$. The following examples illustrate the characterization in Theorem 2(ii).

Example 2 (Quadratic utility). Let $\lambda_1 > \lambda_2$, the utility function be quadratic, $u_i(x) = \theta_i - x$, and increasing, i.e., $\min\{\underline{\theta}_1, \underline{\theta}_2\} \geq 1$. Let $u(\lambda_2; \overline{\theta}_1) = \overline{\theta}_1 - \lambda_2 < \underline{\theta}_2 - 1 + \lambda_2 = u(1 - \lambda_2; \underline{\theta}_2)$, which is a stronger version of Assumption 1; it implies that bidder 1 winning more than λ_2 is never efficient. This assumption guarantees that the constant strategy forms an ex-post equilibrium. Plugging the quadratic utility functions into condition (♠) leads to

$$\overline{\theta}_1 \leq \frac{\underline{\theta}_2(1 - \lambda_2) + \frac{1}{2}\lambda_1^2 - \frac{1}{2} + \lambda_2 - \lambda_2^2}{\lambda_1 - \lambda_2}.$$

The assumption that the efficient share is not too large also places an upper bound on bidder 1's types: $\overline{\theta}_1 < \underline{\theta}_2 - 1 + 2\lambda_2$. Whenever this upper bound is satisfied, condition (♠) also holds as

$$\overline{\theta}_1 < \underline{\theta}_2 - 1 + 2\lambda_2 \leq \frac{\underline{\theta}_2(1 - \lambda_2) + \frac{1}{2}\lambda_1^2 - \frac{1}{2} + \lambda_2 - \lambda_2^2}{\lambda_1 - \lambda_2}.$$

To see that the latter inequality holds, multiplying by the denominator and collecting

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[14]There is a version of the constant strategy that is always an ex-post equilibrium. It involves bidder 2 to "overbid" as a threat to deter bidder 1 seeking to win λ_1. In this strategy, bidder 2 keeps the headline demand at λ_2 until the clock price reaches $p' = U(\lambda_1; \overline{\theta}_1)/\lambda_2$. Bidder 1 then needs to bid at least $p'\lambda_2 = U(\lambda_1; \overline{\theta}_1)$ to win λ_1. Note that bidder 2's threat may not be credible, however, as it may involve bids above the willingness-to-pay. A practical implication is that bidders with tighter caps might want to signal "strength" disproportionately more.

the terms leads to

$$0 \leq (1 - \lambda_1) \left(\underline{\theta}_2 - \frac{1}{2}(1 - \lambda_1) \right) + (\lambda_1 - \lambda_2)^2.$$

The right-hand side is true because $\underline{\theta}_2 \geq 1$ and $0 < \lambda_1 < 1$. ◀

Example 3 (Linear utility). Let $\lambda_1 > \lambda_2$ and the utility function be linear in x, i.e., $u_i(x) = \theta_i$. Condition (♠) reduces to

$$\overline{\theta}_1 \leq \underline{\theta}_2 \frac{1 - \lambda_2}{\lambda_1 - \lambda_2}.$$

If $\lambda_1 = 0.7$ and $\lambda_2 = 0.6$, the highest type of bidder 1 can be up to four times the lowest type of bidder 2 for the constant bidding equilibrium to be sustained. As the caps tend to be as asymmetric as possible ($\lambda_1 \to 1$ and $\lambda_2 \to 1/2$), the equilibrium exists only if bidder 1's highest type is at most (marginally above) bidder 2's lowest type. ◀

The constant strategy features "demand expansion" when the bidders have decreasing marginal values.[15] This may be surprising because one typically associates clock auction and pay-as-bid auction formats with demand reduction rather than demand expansion. Here, the demand expansion leads to an inefficient allocation in which the strong bidder may win an even greater quantity than in the efficient allocation. If there is such bidder asymmetry in the market, one option for the auction designer would be to introduce caps that are tighter on the stronger bidder than on the weaker one. With non-decreasing marginal values, bidder 1 reduces demand compared to truthful bidding while bidder 2 bids "almost clock-truthfully."[16]

The constant strategy also provides an opportunity to raise rivals' costs despite the presence of pay-as-bid pricing in the CMRA.[17] Suppose bidder i knows that the other bidder is stronger and bids according to the constant strategy. A best response for bidder i is to demand $1 - \lambda_2$ at clock price 0. However, following the constant strategy leads to a higher price for bidder j. In particular, bidder i with a lexicographic preference for raising bidder j's payment will follow the constant strategy; extending the clock beyond $p^f(\theta_i)$ leads to the risk of winning λ_2 at clock price $p^f(\theta_j)$.

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[15]Bidder 1 always reduces demand relative to truthful headline demands for low clock prices.

[16]The bidding behavior of bidders with decreasing marginal values can be interpreted as them bidding clock-truthfully under non-decreasing marginal values. Janssen and Kasberger (2019) describe an equilibrium in the CCA in which bidders with decreasing marginal values also act as if their marginal values are non-decreasing (jumps in the demand function).

[17]It is well-known that VCG pricing can provide ample opportunities to raise rival's cost in the CCA (Janssen and Karamychev, 2016; Levin and Skrzypacz, 2016; Janssen and Kasberger, 2019).

6 Collusion via risk-free demand reduction

We now discuss how additional bids can be used to collude via demand reduction in the CMRA. In many dynamic auctions, bidders have incentives to (tacitly) coordinate on low demands in early rounds to close the auction at low prices. However, in a clock auction or a CCA, if a bidder reduces demand, the activity rule (monotonicity constraint) requires demand to stay low. Therefore, (unilateral) demand reduction can be a risky strategy.

In contrast, the CMRA's additional bids can be on small quantities while the headline demand is high. In particular, at low clock prices, bidders can use additional bids on small quantities to find a revenue-maximizing allocation at which a bid from each bidder is accepted. Demand reduction using additional bids is therefore risk-free in the sense that the bidder does not lose the ability to demand large quantities at higher clock prices if the coordination at low clock prices is unsuccessful.[18]

Concretely, we propose the following collusive strategy in the CMRA. Bidders follow the constant strategy as in Theorem 2 but augment it by a single additional bid of 0 for quantity 1/2 in the initial clock round. If both bidders bid this way, the revenue-maximizing allocation immediately assigns half of the supply to each bidder and the revenue is 0. Note that as the headline demand is λ_2, the bidder can later still demand large quantities if the auction does not end in the first round. This makes the demand reduction strategy risk-free for bidders. We call this modified strategy the *risk-free demand reduction* (RDR) strategy. Formally, the headline demand of the RDR strategy is as in Equation (4) and there are two additional bids: $A_i(1/2; 0) = 0$ and $A_i(1 - \lambda_2; p^f(\theta_i)) = 0$.

The next theorem shows that both bidders following the risk-free demand reduction strategy is an ex-post equilibrium in proxy strategies if the bidders are sufficiently similar to each other.[19] The existence of this equilibrium relies on a focal allocation that splits the market and is acceptable to every bidder.[20]

Theorem 3. *Suppose that both bidders have either decreasing or non-decreasing marginal values.*

(i) *Let $\lambda_1 = \lambda_2$. The CMRA has an ex-post equilibrium in proxy strategies in which bidder i chooses the risk-free demand reduction strategy if and only if*

$$U(\lambda_2; \overline{\theta}_i) - U(1/2; \overline{\theta}_i) \leq U(\lambda_2; \underline{\theta}_j) - U(1 - \lambda_2; \underline{\theta}_j) \tag{♣}$$

holds for $i = 1, 2$ and $j = 3 - i$.

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[18]Note that a dominant strategy is truely 'risk-free.' In contrast, our definition of risk-free is not about payoffs, but about the ability to demand large quantities following an unsuccessful attempt to end the auction at low clock prices.

[19]The risk-free demand reduction strategies form a Bayes-Nash equilibrium if the other bidder is sufficiently strong in expectation.

[20]In our model, $(1/2, 1/2)$ is such a focal split. In practice, there might be no such focal allocation or indeed many of them (so bidders would need to use multiple additional bids to reach one).

(ii) *Let $\lambda_1 > \lambda_2$. The CMRA has an ex-post equilibrium in proxy strategies in which bidder i chooses the risk-free demand reduction strategy if conditions (♠) and (♣) hold.*

Using the constant strategy as the continuation limits the potentially profitable deviations. Since the constant strategy profile forms an ex-post (continuation) equilibrium, we essentially only have to check incentive compatibility at clock price 0. In particular, to prove that the additional bid on 1/2 at clock price 0 is incentive compatible, it is sufficient to ask whether a bidder prefers winning 1/2 for free over the outcome when both follow the constant strategy. In the latter case, the stronger bidder i wins λ_2 for transfer $p^f(\theta_j)\lambda_2$ and the weaker bidder j wins $1 - \lambda_2 < 1/2$ for free. Hence, it is apparent that the weaker bidder has no incentive to deviate from the RDR strategy. The stronger bidder also prefers the RDR strategy as

$$U_i(1/2) > U_i(\lambda_2) - p^f(\theta_j)\lambda_2 = U_i(\lambda_2) - \lambda_2 \frac{U_j(\lambda_2) - U_j(1 - \lambda_2)}{\lambda_2},$$

which is implied by Condition (♣). In the symmetric-caps case, the continuation strategies always form an ex-post equilibrium. However, in the asymmetric-caps case, bidder 1 may want to win λ_1 instead of 1/2 for free. To make such a deviation unprofitable, we also impose condition (♠).

Condition (♣) holds when the types are sufficiently close to each other.[21] The following examples illustrate when the RDR equilibrium exists.

Example 4 (Quadratic utility). Let $u(x; \theta) = \theta - x$. For simplicity, we consider ex-ante symmetric bidders with $\lambda_1 = \lambda_2 = \lambda$ and $\underline{\theta}_1 = \underline{\theta}_2 = \underline{\theta}$ and $\overline{\theta}_1 = \overline{\theta}_2 = \overline{\theta}$. We show that for $\underline{\theta} > 1.25$, whenever Assumption 1 is satisfied, so is condition (♣). Under the symmetry assumptions, Assumption 1 reduces to $\overline{\theta} < \underline{\theta} - 1 + 2\lambda$. Condition (♣) reduces to

$$\frac{1}{2}\overline{\theta}(2\lambda - 1) + \frac{1}{8} \le \underline{\theta}(2\lambda - 1) + \frac{(1-\lambda)^2}{2}.$$

Rewriting, we obtain $\overline{\theta} \le 2\underline{\theta} - \frac{3}{4} + \frac{\lambda}{2}$. Assumption 1 implies condition (♣) if

$$\overline{\theta} \le \underline{\theta} - 1 + 2\lambda \le 2\underline{\theta} - \frac{3}{4} + \frac{\lambda}{2}.$$

The latter inequality reduces to $-1/4 + 3\lambda/2 \le \underline{\theta}$. The inequality holds for all λ if $\underline{\theta} > 1.25$.

Example 5 (Linear utility). Let $U(x; \theta) = \theta x$. Condition (♣) boils down to $\underline{\theta}_j(2\lambda_2 - 1) \ge \overline{\theta}_i(2\lambda_2 - 1)/2$ and holds as long as a bidder's strongest type is at most twice the weakest type of the other bidder.

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[21]The RDR strategy can be modified to accommodate more than two bidders: Bidders submit additional bid $A_i(1/n; 0) = 0$ at a clock price of 0 and choose a suitable continuation strategy. Provided that bidders are sufficiently similar, the strategy profile will form an ex-post equilibrium.

6.1 An alternative activity rule to mitigate collusion

We now take a closer look at which part of the auction rules make the demand reduction strategy risk-free and how the rules might be modified to make collusion riskier. Mechanically, the demand reduction in the CMRA works through the additional bids. But it is the activity rule that makes collusion risk-free. Recall that in the RDR strategy the bidder expresses zero marginal value between $1/2$ and λ in the initial clock round. If the auction does not end at $p = 0$, the bidder raises the bid on λ (through the headline demand the bid on λ is $p\lambda$) while keeping the additional bid on $1/2$ constant at 0. Note that the CMRA uses an activity rule in which only headline demands place constraints on the bidding function and only for quantities between two headline demands. But activity rules are often introduced to ensure consistent bidding along the auction price path, usually in the sense of revealed preferences (Ausubel and Baranov, 2014, 2020). However, the activity rule used in the CMRA does not transform additional bids into constraints on the bid function; as a result, observed bids can violate revealed preference.[22]

We therefore propose the following *alternative activity rule* that also transforms the additional bids into constraints.[23] Assuming a finite number of bids, let $X(p)$ denote the set of all quantities bidder i submits non-negative bids on at clock price p: $X(p) = \{x : B_i(x; p) \geq 0\}$. Consider any adjacent $x, x' \in X(p)$ with $x < x'$,[24] and denote $B_i(x'; p) - B_i(x; p) = z$. If $z \geq 0$ and $y \in (x, x']$, then for all $p' \geq p$ the alternative activity rule requires that

$$B_i(y; p') \leq B_i(x; p') + z \frac{y - x}{x' - x}.$$

The alternative activity rule strictly strengthens the activity rule used in the real-world implementations of the CMRA (described in Section 3 and used throughout our analysis). To see this, suppose that bidder i reduces his headline demand from x' to x at clock price p. Then the bidding function is such that $B_i(x'; p) - B_i(x; p) = p(x' - x) = z$. The relative cap (with respect to the headline demands) implies $B_i(y; p') \leq B_i(x; p') + z(y - x)/(x' - x)$ for $x < y < x'$.

Effect on risk-free demand reduction. The alternative activity rule renders the RDR strategy infeasible. Suppose behavior in the first clock round is as according to the RDR strategy: headline-demand λ at clock price $p = 0$ and submit additional bid

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[22]To see this, note that at $p = 0$, the bidder bids $B(1/2; 0) = 0$ and $B(\lambda; 0) = 0$. Hence, the bidder is indifferent between $1/2$ and λ: $U(1/2) = U(\lambda)$. At $p > 0$, the utility-maximizing quantity cannot be λ because $U(\lambda) - p\lambda < U(1/2) - p/2$ and $\lambda > 1/2$.

[23]Our alternative activity rule is deliberately straightforward and we propose it to show clearly how it affects the equilibria we discuss. It is an interesting research direction to consider whether weaker activity rules, such as the GARP activity rule, would also have good properties in our context (Ausubel and Baranov, 2014, 2020).

[24]That is, for any $x'' \in (x, x') \Rightarrow x'' \neq X(p)$.

$A(1/2; 0)$. Hence, the only two quantities that receive non-negative bids are $1/2$ and λ and $B(\lambda; 0) - B(1/2; 0) = 0$. The alternative activity rule then specifies that for $p > 0$ and $y > 1/2$ the bidding function must satisfy $B(y; p) \leq B(1/2; p)$. This constraint conflicts with the RDR strategy, which features $B(\lambda; p) = p\lambda > B(1/2; p) = 0$ for p slightly larger than 0. Moreover, after the initial round behavior of the RDR strategy, the alternative activity rule rules out headline demands above $1/2$ at $p > 0$. To see this, suppose the bidder headline-demands $x > 1/2$. Then the alternative activity rule requires $B(x; p) = px \leq B(1/2; p)$. The auction rules also require $B(1/2; p) \leq p/2$, which leads to a contradiction.

Effect on other collusive strategies.   The alternative activity rule does not, of course, rule out all collusive strategies. In particular, the bidders may attempt to collude in headline demands alone—this is similar to the sort of collusion that is familiar from standard clock auctions (Brusco and Lopomo, 2002; Grimm et al., 2003). However, such collusion is risky because if the headline demand is reduced and the auction does not end, then the bidder is constrained to bidding on lower quantities for the rest of the auction.

Effect on truthful bidding.   The activity rule does not rule out truthful bidding. At low prices, a truthful bidder submits a bid for the maximum desired quantity so the activity rule—which aims to prevent a bidder from submitting low initial bids—has no bite. Moreover, note that CMRA-truthful bidding continuously increases the interval of quantities that receives non-negative bids. For any interval $[a, \lambda_i]$ such that $B(a; p) \geq 0$, the *marginal* bids on $[a, \lambda_i]$ do not change for $p' > p$. Let us revisit the example in Section 2. Each bidder headline-demands 3 units and submits an additional bid of 0 on 2 units at clock price of 10. The corresponding bids are $B(3; 10) = 30$ and $B(2; 10) = 0$, which leads to the constraint $B(3; p) \leq B(2; p) + 30$ for $p > 10$. Note, however, that the additional bid for two units is always $B(2; p) = 3p - 30 = B(3; p) - 30$. Clock-truthful bidding is also not affected by the alternative activity rule.

Effect on the constant strategy.   The alternative activity rule requires the constant strategy to be modified for $p > p_i^f$. One natural modification is to maintain the indifference between the headline demand λ and $1 - \lambda$ by increasing the additional bid on $1 - \lambda$ to $B_i(1 - \lambda; p) = U_i(1 - \lambda) - U_i(\lambda) + p\lambda$. We now show that the modified strategy is feasible. The activity rules requires

$$B_i(\lambda; p) \leq B(1 - \lambda; p) + p_i^f \lambda.$$

Plugging in $B_i(\lambda; p) = p\lambda$ and the (truthful) bid for $1 - \lambda$ reduces the rule to $p_i^f \leq p_i^f$. Hence, the profile of constant strategies can be modified to form an ex-post equilibrium

	Additional bids in bidding data		
	Few early, more later	Many early	None or few
Headline only	NA	NA	Clock-truthful
Headline/additional	CMRA-truthful (non-dec.)	NA	Constant bidding
Additional only	CMRA-truthful (dec.)	RDR	NA

(Winning bids)

Figure 4: Predicted bidding patterns under different bidding strategies and marginal valuations.
Note: "NA" are bidding patterns that cannot be explained by our results. RDR stands for risk-free demand reduction.

under the alternative activity rule.

While the alternative activity rule can mitigate collusion without a substantive impact on other equilibrium strategies, it might affect bidding in ways that our model does not take into account. In particular, bidders need to anticipate the effect of the extra bidding constraints on the ability to submit bids later in the auction and therefore to keep track of (potentially) many bundles as prices evolve. As a result, bidders might be more likely to "miss" bids and find it more difficult to manage their budgets.

7   What theory predicts about bidding in practice

In the previous sections, we have identified four patterns of bidding: clock-truthful, CMRA-truthful, the constant strategy, and risk-free demand reduction. Some of these form equilibria, others do not. In equilibrium there can be demand expansion, demand reduction, and (CMRA-) truthful bidding. We now discuss how theory and auction data can be combined to learn something about bidding behavior.

Figure 4 summarizes which bidding patterns can be explained by the strategies we have described and suggests which bidding patterns in the CMRA would require new theoretical results. While the bidding strategies are inspired by our equilibrium analysis in Sections 5 and 6, our predictions do not rely on equilibrium bidding.

In our model, each bidding pattern we have analyzed has different implications for how the auction ends. If only headline demands are winning, the pattern of bidding is consistent with clock-truthful bidding (or with other clock auction behavior such as standard demand reduction) and we can rule out CMRA-truthful bidding, constant bidding and the risk-free demand reduction strategy. If a mix of headline and additional bids is winning, the pattern of bidding is consistent with CMRA-truthful bids and constant bidding and we can rule out clock-truthful bidding and the risk-free demand reduction strategy. Finally, if only additional bids are winning, the pattern is consistent with the risk-free demand reduction strategy and CMRA-truthful bidding under decreasing marginal values, but we can rule out CMRA-truthful bidding in the case of non-decreasing marginal values, as well as clock-truthful bidding and constant bidding. Of course, we cannot

distinguish between CMRA-truthful bidding under non-decreasing marginal values and constant bidding from winning data only because they are outcome equivalent.

Under CMRA-truthful bidding, a bidder starts with none or few additional bids. In the course of the auction, they not only add new additional bids but also raise existing additional bids. Under the risk-free demand reduction strategy, a bidder starts with a headline demand of a large quantity and additional bids for smaller quantities.[25] This cannot happen under truthful bidding because a bidder cannot be indifferent between a large and small quantity of spectrum at equally low prices. Under risk-free demand reduction, we therefore expect high variance in the bid quantities in early rounds and for the auction to end quickly. Under the constant bidding strategy, the headline demand is constant and additional bids only come late in the auction. The additional bid in this case is low but is made at a relatively high clock price.

While the ideal test of bidding behavior would examine actual bidding data, most telecom regulators are reluctant to publish bidding data in spectrum auctions because it can contain highly sensitive commercial information. Therefore, published auction data is often quite restrictive and do not allow us to unambiguously identify bidding strategies. In the next section we examine Danish CMRAs for which the DEA only publishes aggregated payments and allocations for each bidder. As we will show, despite the coarse data, our theoretical predictions can still be used to assess whether observed outcomes are consistent with the described patterns.

8   The CMRA in Danish spectrum auctions

We now discuss the outcomes of three Danish spectrum auctions that used the CMRA.[26] In each auction there were three bidders whom we refer to as A, B, and C. Each auction allocated lots with and without a coverage obligation. The structure of the auction was such that first the lots with a coverage obligation were allocated (at reserve prices), and then a CMRA was used for the remaining lots. The CMRA first allocated band-specific generic lots. A subsequent assignment phase allocated the specific frequencies. All auctions featured symmetric spectrum caps.

The outcomes changed as bidders became more acquainted with the auction format. In the first auction, it appears that bidders did not use additional bids to end the auction. In the second auction, one bidder basically won the spectrum cap and there is evidence

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[25]In our model, bidders only submit one additional bid in the risk-free collusive strategy, but in practice with many lots across many bands, it might make sense to submit a number of additional bids which would reflect salient ways of splitting the market. Indeed, bidders have many opportunities to coordinate on a collusive market split at clock prices higher than the reserve.

[26]We do not consider the 2020 Norwegian spectrum CMRA. As this auction allocated licenses in an unusually large number of different frequency bands, it is probable that the main task of the auction was achieving coordination among the bidders rather than creating competition for scarce licenses. Excess supply in the auction is evidence for this.

that additional bids were winning. A bidder may have ended the auction with a low additional bid that was made on a small package. In our theoretical framework, this would amount to the constant strategy or to CMRA-truthful bidding with non-decreasing marginal values. In the third auction, there is again evidence that additional bids were winning. The prevalence of additional bids in later auctions suggests that bidders might have become more familiar with the auction format over time.

8.1 Denmark's 1800 MHz spectrum auction

In 2016, the DEA sold licenses in the 1800 MHz band. The bidders were bidding for 2x65 MHz paired frequencies in the 1800 MHz frequency band. The auction first allocated three 2x10 MHz blocks with a coverage obligation non-competitively: Each bidder won a 2x10 MHz block at the reserve price of DKK 50 million.[27] The remaining 2x35 MHz were sold in a CMRA. After assigning the blocks with the coverage obligation, there were spectrum caps that allowed each bidder to win at most 2x20 MHz in the CMRA (which corresponds to a λ of 0.57).

The CMRA allocated seven lots (2x5 MHz blocks). Each bidder was allowed to win at most four blocks in the CMRA. The reserve price was DKK 25 million. The CMRA allocated generic blocks; specific frequencies were allocated in an assignment stage after the CMRA. The assignment stage used a VCG auction.

Bidder A won 2x20 MHz for a total price of DKK 300,159,486. Bidder B won 2x20 MHz for a total price of DKK 300,159,486. Bidder C won 2x25 MHz for a total of DKK 425,239,229. Each bidder won 2x10 MHz with the coverage obligation, so bidders A and B both won two lots in the CMRA. Bidder C won three lots in the CMRA.

It is likely that only headline demands were winning. The first evidence is that bidders A and B paid exactly the same amounts. Moreover, after subtracting the reserve price for the 2x10 MHz with the coverage obligation, the payment per lot in the CMRA was exactly DKK 125,079,743 *for each bidder*. We view this as evidence that the final clock price was DKK 125,079,743. It also suggests that the VCG prices in the assignment stage were 0.

The CMRA ending with headline demand only is only consistent with clock-truthful bidding. While it is probable that bidders did not submit truthful headline demands but reduced their headline demands, there is no evidence that they used additional bids to end the auction as they would in CMRA-truthful bidding, in the constant strategy, or in risk-free demand reduction.

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[27]On 30 September 2016, DKK 1 = \$0.15.

8.2 Denmark's 700 MHz, 900 MHz and 2300 MHz auction

In early 2019, the DEA auctioned licenses for frequencies in the 700 MHz, 900 MHz, and 2.3 GHz bands. The DEA used the CMRA to sell 16 blocks across two bands. Revenue was almost DKK 2.2 billion. A 40 MHz block in the 2.3 GHz band was unsold. We believe that it did not sell due to an associated coverage obligation.

There was a large difference in bidders' payments. In Online Appendix OA.1, we argue it is plausible that bidder B won their headline demand, bidder C won with an additional bid, and bidder A either won their headline demand or with an additional bid. Bidder B won the maximum quantity allowed by the spectrum cap (excluding the unsold lot with the coverage obligation that received no bids), which is consistent in our model with CMRA-truthful bidding under non-decreasing marginal values or with the constant strategy. Bidder C won a small quantity at a relatively low price, which is also consistent with these two strategies.

8.3 Denmark's 2021 auction

In 2021, the DEA allocated frequencies in the 1.5 GHz, 2.1 GHz, 2.3 GHz, 3.5 GHz and 26 GHz bands. Again, the DEA first auctioned off blocks with coverage obligations and then used the CMRA to sell the rest. The DEA used the CMRA to auction a total of 39 blocks across all the five bands. There were three bidders and revenue was just above DKK 2 billion. The auction ended with no lots unsold. If all lots had sold at reserve price, revenue would have been DKK 865 million. In Online Appendix OA.2, we argue that at least some additional bids were winning as linear closing prices seem unlikely. Again, this suggests that bidders were not simply using headline demands.

9 Conclusion

In this paper, we provided the first theoretical analysis of the CMRA, an auction format that has been used in several European spectrum sales. Our analysis of strategic bidding suggests that truthtelling equilibria are vulnerable to bidder asymmetries and that inefficient equilibria might emerge. Crucially, however, the CMRA is prone to risk-free collusive bidding when bidders are sufficiently symmetric. This paper therefore underlines the importance of rigorous theoretical analysis of auction formats in the process of auction design.

There are several avenues for further work. First, one could develop the theoretical analysis by looking at other equilibria in our model or by extending the model to heterogeneous goods. Second, one could take our theoretical predictions to real-world or experimental bidding data.[28] Third, one could analyze bidder incentives under richer

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[28]Some combinatorial auctions have been studied in laboratory experiments (e.g., Kagel et al., 2010,

utility functions that include, for example, spitefulness or risk aversion. Finally, one could explore other alternative activity rules and the associated trade-offs in this auction format.

A Omitted Proofs

A.1 Proofs of Section 4

Proof of Proposition 1. We prove each of the three statements separately.

1. The truthful headline demands are continuous and decreasing in p. Thus, the auction ends efficiently and with market clearing at price $p^\star$.

2. We first show that CMRA-truthful bidding cannot end in an inefficient allocation. Let x be an inefficient allocation in which bidder i wins, wlog, less than in the efficient allocation ($x_i < x_i^\star$). The allocation x can only be revenue-maximizing if both bidders have bid on their shares. Then bidder i's marginal bid is higher than bidder j's marginal bid. Thus it is profitable to allocate more to bidder i.

   In any round p, the revenue-maximizing allocation is either the efficient allocation or a boundary allocation (($\lambda_i, 0$) for low prices or ($\lambda_i, 1 - \lambda_i$) for high prices). Note that $1 - \lambda_j < x_i^\star$. Hence, there are prices p for which bidder i bids on $x_i^\star$ but not on $1 - \lambda_j$. For prices p for which at least one bidder bids on $1 - \lambda_i$, we have

   $$B_i(x_i^\star; p) + B_j(x_j^\star; p) = U_i(x_i^\star) - V_i(p) + U_j(x_j^\star) - V_j(p) > B_i(\lambda_i; p) + B_j(1 - \lambda_i; p).$$

   It follows that ($\lambda_i, 1 - \lambda_i$) cannot be the final allocation; the auction ends before one bidder i bids on $1 - \lambda_j$. The CMRA then ends at the lowest price $\tilde{p}^\star$ such that

   $$B_i(x_i^\star; \tilde{p}^\star) + B_j(x_j^\star; \tilde{p}^\star) = \max_{i=1,2} B_i(\lambda_i; \tilde{p}^\star);$$

   the auction ends as soon as the interior (efficient) revenue-maximizing allocation is a global maximum of revenue. We have $\tilde{p}^\star < p^\star$ as

   $$B_i(x_i^\star; p^\star) + B_j(x_j^\star; p^\star) = p^\star > p^\star \lambda_i > B_i(\lambda_i; p^\star) = U_i(\lambda_i) - U_i(x_i^\star) + p^\star x_i^\star$$

   due to decreasing marginal values (bids) and truthful headline demands. The inequality implies that at $p^\star$, the unique revenue-maximizing allocation is the efficient allocation, so the auction must have ended at a lower price.

2014; Bichler et al., 2013).

3. We now show that revenue is lower. Recall that the CMRA ends at $\tilde{p}^\star < p^\star$. Bidder i's contribution to revenue is

$$B_i(x^\star; \tilde{p}^\star) = U_i(x^\star) - V_i(\tilde{p}^\star) \le x^\star \tilde{p}^\star < x^\star p^\star.$$

The first inequality is true as $x^\star$ is not the utility-maximizing demand at $\tilde{p}^\star$.

$\square$

Proof of Proposition 2. We prove each of the three statements separately.

1. The clock-truthful demand is λ_i until the price reaches $U_i(\lambda_i)/\lambda_i$. For higher prices demand equals 0. Hence, the clock ends at price $\min U_i(\lambda_i)/\lambda_i$ and it does so with excess supply. Due to positive marginal values for all shares below λ_i, the efficient allocation does not feature excess supply. The outcome is inefficient.

2. Consider CMRA-truthful bidding. For $p \le U_i(\lambda_i)/\lambda_i$, truthful additional bids are $B_i(x; p) = U_i(x) - V_i(p)$, where $V_i(p) = U_i(\lambda_i) - p\lambda_i$. At price p_i^f bidder i is indifferent between winning λ_i for a payment $\lambda_i p_i^f$ and winning $1 - \lambda_i$ for free. This leads to
$$p_i^f = \frac{U_i(\lambda_i) - U_i(1 - \lambda_j)}{\lambda_i}.$$

Hence, the auction may end at clock price $\min\{p_1^f, p_2^f\}$ with market clearing. Note, however, that if $p_1^f < p_2^f$ and $\lambda_1 > \lambda_2$, then the allocation $(1 - \lambda_2, \lambda_2)$ is not revenue-maximizing at clock price p_1^f despite receiving non-negative bids from both bidders; this is because $\lambda_1 p_1^f > \lambda_2 p_1^f + 0$. In the following three exhaustive cases, we show that the auction ends nevertheless with the efficient allocation.

Case 1 ($p_2^f \le p_1^f$). The CMRA ends at clock-price p_2^f with the allocation $(\lambda_1, 1 - \lambda_1)$ as this allocation receives bids from both bidders and is revenue-maximizing. Suppose the outcome were inefficient, i.e., $U_1(1 - \lambda_2) + U_2(\lambda_2) > U_1(\lambda_1) + U_2(1 - \lambda_1)$. Then we observe that

$$p_2^f = \frac{U_2(\lambda_2) - U_2(1 - \lambda_1)}{\lambda_2} > \frac{U_1(\lambda_1) - U_1(1 - \lambda_2)}{\lambda_2} \ge \frac{U_1(\lambda_1) - U_1(1 - \lambda_2)}{\lambda_1} = p_1^f,$$

which is a contradiction to $p_2^f \le p_1^f$.

Case 2 ($p_1^f < p_2^f$ and $(\lambda_1, 1 - \lambda_1)$ is efficient). Bidder 1 bids zero on $1 - \lambda_2$ at clock price p_1^f so the allocation $(1 - \lambda_2, \lambda_2)$ receives bids from both players. The allocation is not revenue-maximizing, however, if $\lambda_1 > \lambda_2$. In this case, the CMRA continues. The price at which the auction may end with allocation $(1 - \lambda_2, \lambda_2)$ is such that

$$U_1(1 - \lambda_2) - U_1(\lambda_1) + p\lambda_1 + p\lambda_2 = p\lambda_1.$$

The price that solves the equality is

$$p = \frac{U_1(\lambda_1) - U_1(1 - \lambda_2)}{\lambda_2},$$

which is larger than p_1^f. Moreover, as $(\lambda_1, 1 - \lambda_1)$ is efficient, this price is larger than p_2^f. Consequently, bidder 2 bids 0 on $1 - \lambda_1$ before the joint bid on allocation $(1 - \lambda_2, \lambda_2)$ exceeds $p\lambda_1$. The auction ends at p_2^f with the efficient allocation.

Case 3 ($p_1^f < p_2^f$ and $(1 - \lambda_2, \lambda_2)$ is efficient). As $(1 - \lambda_2, \lambda_2)$ is efficient, the prices satisfy

$$p_1^f = \frac{U_1(\lambda_1) - U_1(1 - \lambda_2)}{\lambda_1} < \frac{U_1(\lambda_1) - U_1(1 - \lambda_2)}{\lambda_2} < \frac{U_2(\lambda_2) - U_2(1 - \lambda_1)}{\lambda_2} = p_2^f.$$

At p_1^f, the efficient allocation receives bids from both bidders; however, the joint bid on it is not revenue-maximizing for prices slightly above p_1^f. This changes at $p = \frac{U_1(\lambda_1) - U_1(1 - \lambda_2)}{\lambda_2}$. The auction does not end earlier with allocation $(\lambda_1, 1 - \lambda_1)$ as bidder 2 does not bid on $1 - \lambda_1$.

3. Let $\lambda_1 = \lambda_2 = \lambda$. Under clock-truthful bidding, the clock ends at clock price $\min_i U_i(\lambda)/\lambda$. Under CMRA-truthful bidding the auction ends at a lower clock price, namely $\min_i p_i^f$. As ex-post revenue is λ times the final clock price in both cases, revenue is lower under CMRA-truthful bidding due to the lower final clock price.

$\square$

A.2 Proofs of Section 5

Proof of Proposition 3. Suppose both bidders bid CMRA-truthfully. The auction ends at price $\tilde{p}^\star$ such that

$$B_1(x_1^\star; \tilde{p}^\star) + B_2(x_2^\star; \tilde{p}^\star) = \max_i B_i(\lambda_i; \tilde{p}^\star).$$

The bidder with the higher bid on λ_i has an incentive to stop raising bids on quantities that can no longer be efficient shares. $\square$

Proof of Theorem 1. Let $\lambda_1 = \lambda_2 = \lambda$. Suppose bidder j bids CMRA-truthfully. If bidder i also bids in such a way and $\theta_i \geq \theta_j$, then the auction ends with bidder i winning λ for a payment $p_j^f \lambda$. If $\theta_i < \theta_j$, then bidder i wins $1 - \lambda$ for free. In the first case the surplus is $U_i(\lambda) - U_j(\lambda) + U_j(1 - \lambda)$. In the second case the surplus is $U_i(1 - \lambda)$.

Suppose bidder i wants to win x for payment $D_i(x)$. As bidder j's headline demand is λ, bidder i's bid $D_i(x)$ at price p is winning only if

$$D_i(x) + B_j(1 - x; p) \geq B_j(\lambda; p).$$

Ideally, this inequality is binding as otherwise the bid $D_i(x)$ can be decreased. Plugging in the expressions for bidder j's bids yields

$$D_i(x) = U_j(\lambda) - U_j(1 - x).$$

Consider $\theta_i \geq \theta_j$. The deviation is profitable only if the surplus from winning x is better than winning λ, i.e.,

$$U_i(x) - D_i(x) \geq U_i(\lambda) - p_j^f \lambda.$$

Plugging in the expression for $D_i(x)$ gives

$$U_i(x) - U_j(\lambda) + U_j(1 - x) \geq U_i(\lambda) - U_j(\lambda) + U_j(1 - \lambda).$$

Note that this is a (weak) contradiction as it is efficient that bidder i wins λ (strict if $x \neq \lambda$). We conclude that bidder i does not have an incentive to deviate.

Now consider $\theta_i < \theta_j$. The deviation is profitable if

$$U_i(x) - U_j(\lambda) + U_j(1 - x) \geq U_i(1 - \lambda),$$

where the left-hand side is the expected utility of winning x for a payment $D_i(x)$ and the right-hand side is the expected utility from following the CMRA-truthful strategy. The only x for which the inequality holds is $x = 1 - \lambda$ as bidder j winning λ is efficient. Hence, there is no profitable deviation.

Let the caps be asymmetric. We first prove two lemmas.

Lemma A.1. *Let marginal values be non-decreasing and let $\lambda_1 > \lambda_2$. CMRA-truthful bidding forms an ex-post equilibrium in proxy strategies only if $p_2^f \leq p_1^f$ for all feasible type profiles.*

Proof of Lemma A.1. First, suppose $p_1^f < p_2^f$ and $(\lambda_1, 1 - \lambda_1)$ is efficient. Case 2 in the proof of Proposition 2 shows that the auction ends at clock price p_2^f with the efficient allocation. However, bidder 1 prefers winning $1 - \lambda_2$ for free over λ_1 for transfer $p_2^f \lambda_1$:

$$U_1(1 - \lambda_2) - 0 > U_1(\lambda_1) - p_2^f \lambda_1,$$

which is equivalent to $p_1^f < p_2^f$. Second, consider type profiles so that $p_1^f < p_2^f$ and

$(1 - \lambda_2, \lambda_2)$ is efficient. Case 3 in the proof of Proposition 2 implies that the clock ends at price $p = (U_1(\lambda_1) - U_1(1 - \lambda_2))/\lambda_2$ with the efficient allocation, and bidder 1 pays $p(1 - \lambda_2)$. Bidder 1 could have won the same share for free by submitting an additional bid of zero for $1 - \lambda_2$ at clock price 0. In both cases, bidder 1 has a profitable deviation, so CMRA-truthful bidding forms an equilibrium only if $p_2^f \leq p_1^f$. $\qquad\square$

Lemma A.2. *Let marginal values be non-decreasing and let $\lambda_1 > \lambda_2$. Let the bidders' types be such that $p_2^f < p_1^f$. If bidder 2 bids CMRA-truthfully, then bidder 1 has no incentive to bid CMRA-truthfully.*

Proof of Lemma A.2. If both bidders bid CMRA-truthfully, then the auction ends at clock price p_2^f with the efficient allocation $(\lambda_1, 1 - \lambda_1)$. Bidder 1's total payment is $p_2^f \lambda_1$.

Bidder 1 has the following profitable deviation: Headline-demand λ_1 until the clock price reaches $p \in ((U_2(\lambda_2) - U_2(1 - \lambda_1))/\lambda_2, p_2^f)$, at which point the headline demand is dropped to 0. No further bids are made by bidder 1. Hence, $B_1(\lambda_1; p') = p\lambda_1$ for $p' \geq p$.

If bidder 1 plays this strategy, the auction does not end at clock price p, because there is no revenue-maximizing allocations (λ_1, y), where $0 \leq y \leq 1 - \lambda_1$, that has received non-negative bids from both bidders; the auction continues until p_2^f.

The deviation is profitable only if the auction does indeed lead to allocation $(\lambda_1, 1-\lambda_1)$ at clock price p_2^f; allocation $(\lambda_1, 1 - \lambda_1)$ must be revenue-maximizing:

$$\underbrace{B_1(0; p_2^f)}_{=0} + \underbrace{B_2(\lambda_2; p_2^f)}_{p_2^f \lambda_2} < \underbrace{B_1(\lambda_1; p_2^f)}_{=p\lambda_1} + \underbrace{B_2(1 - \lambda_1; p_2^f)}_{=0}.$$

Therefore, for p such that $p_2^f > p > \frac{U_2(\lambda_2) - U_2(1-\lambda_1)}{\lambda_1}$, the deviation does not change the final allocation compared to truthful bidding but reduces bidder 1's total payment from $p_2^f \lambda_1$ to $p\lambda_1$. $\qquad\square$

Due to marginal utilities being strictly increasing in θ, there is no type space (with at least three distinct types) so that $p_1^f = p_2^f$ for all type profiles. To see this, note that p_i^f only depends on i's type. Consider a type profile such that $p_1^f = p_2^f$. Changing θ_i while keeping θ_j constant leads to $p_1^f < p_2^f$ or $p_1^f > p_2^f$. In the first case, Lemma A.1 implies that truthful bidding does not form an equilibrium. In the second case, Lemma A.2 implies that bidder 1 can profitably deviate from truthful bidding. $\qquad\square$

Proof of Theorem 2. The first observation is that $\theta_i \geq \theta_j \Leftrightarrow p^f(\theta_i) \geq p^f(\theta_j)$. To see this, note that $p^f(\theta) = (U(\lambda_2; \theta) - U(1 - \lambda_2; \theta))/\lambda_2 = \int_{1-\lambda_2}^{\lambda_2} u(x; \theta) dx / \lambda_2$. The assumption $du/d\theta > 0$ implies that p^f increases in θ. Moreover, $p^f(\theta) < U(\lambda_2; \theta)/\lambda_2$ for all feasible θ.

Now suppose that both bidders follow the constant strategy with headline demand in Equation (4). Bidder i wins λ_2 for transfer $p^f(\theta_j)\lambda_2$ if $\theta_i \geq \theta_j$ and wins $1 - \lambda_2$ for free if $\theta_i < \theta_j$. The proof follows similar steps as the proof of Theorem 1.

Suppose that instead of following the constant strategy, bidder i wants to win x_i with bid $D_i(x_i)$ at clock price p. As bidder j's headline demand is λ_2 for any clock price below $U_j(\lambda_2)/\lambda_2$, bidder j's bid on λ_2 is $B_j(\lambda_2; p) = \min\{p, U_j(\lambda_2)/\lambda_2\}\lambda_2$. To be part of the revenue-maximizing allocation, the bid $D_i(x_i)$ must satisfy

$$D_i(x_i) + B_j(x_j; p) \geq B_j(\lambda_2; p),$$

where $x_i + x_j \leq 1$.

Consider $p < p^f(\theta_j)$. As bidder j's headline demand is λ_2 and bidder j places no additional bids, bidder i faces

$$B_j(x_j; p) = \begin{cases} -\infty & \text{for } 0 \leq x_j < \lambda_2 \\ p\lambda_2 & \text{for } x = \lambda_2. \end{cases}$$

Hence, the only feasible x_i is $x_i \leq 1 - \lambda_2$. Any such x_i can be won for free. The best x_i is $1 - \lambda_2$. If $\theta_i < \theta_j$, then bidder i cannot do better by deviating from the constant strategy. If $\theta_i \geq \theta_j$, then bidder i prefers the constant strategy as this leads to a higher expected utility:

$$U_i(\lambda_2) - p^f(\theta_j)\lambda_2 \geq U_i(1 - \lambda_2) \Leftrightarrow p^f(\theta_i) \geq p^f(\theta_j).$$

Consider $p \in [p^f(\theta_j), U_j(\lambda_2)/\lambda_2)$. At clock price $p^f(\theta_j)$, bidder j adds the additional bid on $1 - \lambda_2$, which changes the bid function at $1 - \lambda_2$ to $B_j(1 - \lambda_2; p) = 0$. Now the revenue-maximizing allocation can be $(x_i, 1 - \lambda_2)$ for $x_i \leq \lambda_2$ if $D_i(x_i) \geq B_j(\lambda_2; p)$. In the best case, the inequality is binding. Hence, if $D_i(x_i)$ is winning, then bidder i's utility is at most $U_i(x_i) - B_j(\lambda_2; p)$. The x_i that maximizes the utility is $x_i = \lambda_2$. If $\theta_i \geq \theta_j$, then bidder i wins λ_2 for $p^f(\theta_j)\lambda_2$, which is at least as good as winning λ_2 for price $B_j(\lambda_2; p) = p\lambda_2 \geq p_j^f\lambda_2$. Hence, there is no profitable deviation when bidder i is stronger. If bidder i is weaker, then the utility of winning λ_2 for $B_j(\lambda_2; p)$ is less than winning $1 - \lambda_2$ for free, which can be achieved by the constant strategy. Hence, there is also no profitable deviation if bidder i is weaker than bidder j.

Finally, consider $p \geq U_j(\lambda_2)/\lambda_2$. Bidder j bids

$$B_j(x_j; p) = \begin{cases} 0 & \text{for } x_j \in \{0, 1 - \lambda_2\} \\ U_j(\lambda_2) & \text{for } x = \lambda_2 \\ -\infty & \text{else.} \end{cases}$$

Winning $x_i \leq \lambda_2$ now becomes more expensive compared to the previous case. Bidder 1 might also win $x \in (\lambda_2, \lambda_1]$ at clock price p. The lowest (and only) clock price at which this is possible is $p = U_2(\lambda_2)/\lambda_2$. At this price, bidder 2 bids 0 on 0 so that an allocation

in which bidder 1 wins more than λ_2 can involve bids by both bidders. The deviating bid must be sufficiently high so that it is revenue-maximizing:

$$D_1(x) + B_2(0; p) \geq B_2(\lambda_2; p) = U_2(\lambda_2).$$

It follows that bidder 1 can win x for payment $U_2(\lambda_2)$. A strategy that leads to this outcome is to headline-demand x until clock price $\tilde{p} = U_2(\lambda_2)/x$ and then stop bidding. Note that this deviation is the same as the one that proves that CMRA-truthful bidding is not an equilibrium with non-decreasing marginal values (Theorem 1). Due to positive marginal values and a payment that is independent of the quantity, the optimal quantity is $x = \lambda_1$.

Bidder 1's deviation payoff is then $U_1(\lambda_1) - U_2(\lambda_2)$. The deviation is profitable if

$$U_1(\lambda_1) - U_2(\lambda_2) \geq U_1(\lambda_2) - U_2(\lambda_2) + U_2(1 - \lambda_2).$$

Due to marginal values being increasing in θ, if the highest type of bidder 1, $\overline{\theta}_1$, has no incentive to deviate against the lowest type of bidder 2, $\underline{\theta}_2$, then no type of bidder 1 has an incentive to deviate. It follows that condition (♠) is sufficient for the profile of constant strategies to form an ex-post equilibrium.

Moreover, condition (♠) is also necessary because otherwise there are type profiles so that bidder 1 has an incentive to deviate along the lines discussed above.

With decreasing marginal values, the efficient allocation is interior. In contrast, the constant strategy profile leads to allocation $(\lambda_2, 1 - \lambda_2)$, which cannot be efficient by Assumption 1 for all type profiles.

If Assumption 2 holds and $(\lambda_1, 1 - \lambda_1)$ is efficient, then the equilibrium outcome is inefficient. If $(1 - \lambda_2, \lambda_2)$ is efficient, then $\int_{1-\lambda_1}^{\lambda_2} u_2(x)dx \geq \int_{1-\lambda_2}^{\lambda_1} u_1(x)dx$. As both integrals are over the same mass, but over lower x for bidder 2, bidder 2 must have a higher θ. Hence, the profile of constant strategies leads to the efficient outcome. With symmetric caps and with non-decreasing marginal values, the outcome is identical to the outcome under CMRA-truthful bidding and to truthful bidding in a VCG auction. □

A.3   Proofs of Section 6

Proof of Theorem 3. If bidder i does not want to win $1/2$ at clock price 0, then the profile of continuation strategies forms an ex-post equilibrium in proxy strategies by Theorem 2. Hence, the only question is whether bidder i prefers winning $1/2$ for free over the continuation payoff of the constant strategy profile.

Let $\theta_i \geq \theta_j$. Then bidder i wins λ_2 for transfer $U_j(\lambda_2) - U_j(1 - \lambda_2)$ by deviating from

winning 1/2 for free in the initial clock round. The deviation payoff is worse as:

$$U_i(\lambda_2) - U_i(1/2) \leq U(\lambda_2; \overline{\theta}_i) - U(1/2; \overline{\theta}_i)$$
$$\leq U(\lambda_2; \underline{\theta}_j) - U(1 - \lambda_2; \underline{\theta}_j) \leq U_j(\lambda_2) - U_j(1 - \lambda_2).$$

The first and the last inequalities use that marginal values are strictly increasing in type. The second inequality uses condition (♣).

Conversely, if condition (♣) does not hold, then the highest type facing the lowest competitor's type strictly prefers to win λ_2 for transfer $p^f(\underline{\theta}_j)\lambda_2$ over winning 1/2 for free.

Let $\theta_i < \theta_j$. Sticking to the RDR strategy gives bidder i a quantity of 1/2 for free. The deviation of not implementing the 50-50 split leads to a payoff of $1 - \lambda_2$ for free. There is no incentive to deviate. □

B   More than two bidders

We now analyze the robustness of the previous results with respect to the number of bidders n. We start by looking at CMRA-truthful bidding and non-decreasing marginal values because we know that truthful bidding is not an equilibrium with decreasing marginal values (Proposition 3). Moreover, we restrict attention to symmetric caps because we know that CMRA-truthful bidding is not an equilibrium with asymmetric caps, even in the two-bidder case (Theorem 1). Let $1/n < \lambda < 1$.

With non-decreasing marginal values, there are $m = \lceil 1/\lambda \rceil$ winners in the efficient allocation ($\lceil x \rceil$ denotes the smallest integer that is larger than x). In the efficient allocation, $m - 1$ bidders win λ and one bidder wins $1 - (m - 1)\lambda =: \hat{x}$. The other bidders win nothing. Wlog, let us rank the bidders according to type so that bidder 1 has the highest and bidder n has the lowest type. Then bidder i with $i < m$ wins λ and bidder m wins $\hat{x}$; $\hat{x}$ is equal to λ if $\lambda = 1/m$. Bidder $i > m$ does not win anything in the efficient allocation.

The auction ends with the efficient allocation when all bidders bid CMRA-truthfully. The bidding functions B_i are convex and such that $B_i(\lambda) = \min\{p\lambda, U_i(\lambda)\}$. For any lower quantity, lower types bid higher (unless they have bid zero on zero quantity). In particular, low types bid on zero quantity or the residual $\hat{x}$ at lower clock prices. On the other hand, bidders bid true marginal values so that higher types have steeper bidding functions. It follows that the auction ends efficiently: Because low types bid on low quantities earlier and revenue is maximized by assigning the quantity λ to as many bidders as possible, it cannot be that high types win zero or $\hat{x}$.

Let $p_i^f(x)$ denote the clock price at which bidder i bids zero on x under CMRA-truthful

bidding:

$$p_i^f(x) = \frac{U_i(\lambda) - U_i(x)}{\lambda}.$$

Let p^F denote the clock price at which the auction ends.

When all bidders are winners in the efficient allocation (i.e., if $n = m$ or if $1/n < \lambda < 1/(n-1)$), then the clock ends at $p^F = p_m^f(\hat{x})$. This case is analogous to the two-bidder case studied above. When $n > m$, then the auction does not end at $p_m^f(\hat{x})$. At $p_m^f(\hat{x})$, bidder m bids 0 on $\hat{x}$ but bidder $m+1$ bids something strictly positive on $\hat{x}$. Hence, the efficient allocation is not revenue-maximizing and the revenue-maximizing allocation does not include bids by all bidders; the auction continues. At clock price $p_{m+1}^f(0) = U_{m+1}(\lambda)/\lambda$, bidder $m+1$ stops bidding. However, bidder $m+1$'s bid on $\hat{x}$ is still higher than bidder m's until the clock price is

$$p^F = \frac{U_{m+1}(\hat{x}) - U_m(\hat{x}) + U_m(\lambda)}{\lambda},$$

which is such that

$$B_m(\hat{x}; p^F) = U_{m+1}(\hat{x}).$$

Note that this price satisfies $p^F > p_m^f(\hat{x})$ and $p^F \leq p_m^f(0)$. Moreover, $p^F > p_{m+1}^f(0)$ because marginal values are increasing in θ. Importantly, bidders pay VCG prices. Bidder m bids $U_{m+1}(\hat{x})$ on $\hat{x}$, which is the externality m imposes on $m+1$ (and the other losing bidders). Bidder $i \in \{1, 2, \ldots, m-1\}$ pays $p^F \lambda = U_{m+1}(\hat{x}) - U_m(\hat{x}) + U_m(\lambda)$ (which is the externality of bidder i having a higher type so that bidder m does not win λ and bidder $m+1$ does not win $\hat{x}$).

Theorem B.1. *Let there be $n \geq 2$ bidders with symmetric caps $\lambda \in (1/n, 1)$ and non-decreasing marginal values. Then CMRA-truthful bidding is an efficient ex-post equilibrium.*

Proof. We have already argued that CMRA-truthful bidding leads to the efficient allocation. We now prove that no bidder has an incentive to deviate from truthful bidding even if they know the other bidders' types.

Case 1. Let $n = m$. As we argued above, under CMRA-truthful bidding, the auction stops at $p^F = \min_i p_i^f(\hat{x}) = p_m^f(\hat{x})$.

Suppose bidder i wants to win x_i at $p \leq p^F$. If $x_i \leq \hat{x}$, then this quantity can be won for free but

$$U_i(x_i) \leq U_i(\hat{x}) \leq U_i(\lambda) - p\lambda$$

since $p \leq p_i^f(\hat{x})$. Note that share λ can only be won at $p_m^f(\hat{x})$ (or at higher prices if $i = m$). If $\hat{x} < x_i < \lambda$, then the revenue-maximizing allocation can only be such that bidder i wins x_i, $n - 2$ bidders win λ and bidder m wins $1 - x_i - (n-2)\lambda > \hat{x}$ because low types bid higher than high types (except that all bidders bid $p\lambda$ for λ as long as

$p \leq p_n^f(0)$). Hence, bidder i needs to bid at least $D_i(x_i)$ to win x_i in clock round p, where $D_i(x)$ must satisfy

$$D_i(x_i) + B_m(1 - x_i - (n-2)\lambda) + (n-2)\lambda \geq (n-1)p\lambda.$$

The right-hand side contains the revenue when all bidders except for bidder i win λ, which is a feasible allocation but not one in which one bid by every bidder is accepted. The inequality reduces to $D_i(x_i) \geq U_m(\lambda) - U_m(1 - x_i - (n-2)\lambda)$. The optimal deviation is such that bidder i pays the VCG price, i.e., the externality imposed on bidder m. Expected utility is then $U_i(x_i) - U_m(\lambda) + U_m(1 - x_i - (n-2)\lambda)$, which is convex in x_i and maximized by $x_i = \lambda$.

Suppose bidder i wants to win x_i at $p > p^F$. Buying less than $\hat{x}$ is not profitable because this quantity can be obtained for free but it is not preferred to the outcome of CMRA-truthful bidding by construction of the strategy. Buying λ only becomes increasingly costly. Winning $x_i \in (\hat{x}, \lambda)$ is possible even at $p \leq p^F$ but not profitable.

Case 2. Let $n > m$. Note that a deviation by bidder i changes the outcome of at most two other bidders; their identify depends on the deviating bidder's.

Consider first bidder $i = m$. If bidder i wants to win less than $\hat{x}$, then bidder $m+1$ wins more; the other bidders are not affected. Suppose bidder i wants to win x_i less by bidding $D_i(\hat{x} - x_i; p)$ in clock round p. This deviating bid is part of a revenue-maximizing allocation only if

$$D_i(\hat{x} - x_i; p) + B_{m+1}(x_i) \geq B_{m+1}(\hat{x}).$$

Bidder $m+1$ bidding truthfully implies that the deviating bid must be at least $U_{m+1}(\hat{x}) - U_{m+1}(x_i)$. Since this is the VCG price, bidder i has no incentive to deviate from truthful bidding, i.e., $0 \in \arg\max_{x_i} U_i(\hat{x} - x_i) - U_{m+1}(\hat{x}) + U_{m+1}(x_i)$. Suppose bidder $i = m$ wants to win x_i more, so $\hat{x} + x_i$ in total. This can only be done when bidder $m-1$ wins $\lambda - x_i$. Then the deviating bid is part of a revenue-maximizing allocation only if

$$D_i(\hat{x} + x_i; p) + B_{m-1}(\lambda - x_i; p) \geq B_{m-1}(\lambda; p).$$

The deviating bid must be at least the VCG price $U_{m-1}(\lambda) - U_{m-1}(\lambda - x_i)$; it is not profitable to win more. Note that it cannot be the case that bidder i wins so much more that also bidder $m-2$ is affected. If bidder m wins x_i more that $\hat{x}$, than the cap requires $\hat{x} + x_i \leq \lambda$. Hence, even if bidder m wins λ, bidder $m-1$ wins $\hat{x}$ and bidder $m-2$ wins λ.

Consider bidder $i > m$, that is, a bidder who does not win anything if bidding CMRA-truthfully. Such a bidder i can only win more than in the efficient allocation by deviating. Depending on how much bidder i wants to win, this reduces how much bidder m wins or it can reduce what bidders m and $m-1$ win. In the first case, the deviating bid must be

at least $U_m(\hat{x}) - U_m(\hat{x} - x_i)$, where x_i is the amount bidder i wants to win. Since this is the VCG price, there is no profitable deviation; the optimal x_i is 0. In the second case, bidder m does not win anything after the deviation so that the deviating bid must be at least $U_{m-1}(\lambda) - U_{m-1}(\lambda - x_i) + U_m(\hat{x})$; the externality imposed on the other bidders. None of the two possible deviations can be profitable.

Consider bidder $i < m$. Bidder i can only win less than in the efficient allocation due to the cap. Depending on how much bidder i wants to win less, there are two possible cases. First, it can be that only bidder m wins more, and, second, that bidders m and $m+1$ win more. In the first case, bidder i has to bid at least $U_m(\lambda) - U_m(\hat{x} + x_i)$, where x_i is the amount bidder i wants to win less. Maximizing expected utility $U_i(\lambda - x_i) - U_m(\lambda) + U_m(\hat{x} + x_i)$ with respect to x_i leads to $x_i = 0$ due to non-decreasing marginal values. Second, suppose bidder i wishes to win so much less that bidder $m+1$ becomes a winner. The deviating bid must then be such that

$$D_i(\lambda - x_i; p) + B_m(\lambda; p) + B_{m+1}(\hat{x} - \lambda + x_i; p) \geq B_m(\lambda; p) + B_{m+1}(\hat{x}; p).$$

The deviating bid must therefore be at least $U_{m+1}(\hat{x}) - U_{m+1}(\hat{x} - \lambda + x_i)$. It is not profitable for bidder i to deviate. $\qquad\square$

Finally, we check whether the constant strategy forms an equilibrium. We have already noted that the $n = m$ case is essentially as in in the two-bidder case. Hence, the profile of constant strategies forms an ex-post equilibrium if $n = m$ for decreasing and non-decreasing marginal values (with symmetric caps). However, if $n > m$, then bidder $m+1$ has a profitable deviation. Instead of submitting a single bid on $\hat{x}$ with value 0, it is optimal to bid ϵ on $\hat{x}$. Doing so guarantees that bidder $m+1$ rather than bidder m wins $\hat{x}$. Hence, the profile of constant strategies does not form an ex-post equilibrium if $n > m$.

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ONLINE APPENDIX

The Combinatorial Multi-Round Ascending Auction

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OA.1 Details on the 2019 Danish auction

Table OA.1 summarizes the supply in the 2019 auction and the auction outcome. The licenses in the 900 MHz band were not allocated through a CMRA. In the 900 MHz band, there were 2x30 MHz paired frequencies available. These licenses came with a coverage obligation. We call the lots in the 900 MHz band A lots.

The supply in the CMRA was six 2x5 MHz blocks (B lots) of paired frequencies in the 700 MHz band, four 5 MHz blocks of unpaired frequencies in the 700 MHz band (D lots), one block 40 MHz block in the 2.3 GHz band with a coverage obligation (E lot), and six 10 MHz blocks in the 2.3 GHz band (F lots).

There was no reserve price on the lots with a coverage obligation. The reserve price per B lot was DKK 95 million. The reserve price per D lot was DKK 25 million. The reserve price per F lot was DKK 25 million.

The following spectrum caps were in place. Each bidder was allowed to win at most one block in the 900 MHz band. Across the paired blocks in the 700 MHz and 900 MHz bands, each bidder was allowed to win at most four lots. Bidders were not allowed to win more than 60 MHz in the 2.3 GHz band. There was no restriction on the number of blocks a bidder could win in the unpaired 700 MHz band.

The auction outcome was as follows. Bidder A paid DKK 485.2 million for one A lot and two B lots. Bidder B paid DKK 1620 million for one A lot, three B lots, four D lots, and six F lots. Bidder C paid DKK 107.6 million for one A lot and one B lot. Hence, the 40 MHz lot in the 2.3 GHz spectrum with the coverage obligation was unsold. Note that bidder B received the maximum quantity permitted by the spectrum caps, which is consistent with the CMRA-truthful and constant strategies. For this reason, it is likely that bidder B won with their headline demand.

We now examine whether the bidders paid linear prices. Recall this would suggest that only headline demands were winning. As the A lots were traded at a reserve price of 0, bidder C paid DKK 107.6 million for a single B lot. Bidder A paid DKK 485.2 million for two B lots. As bidder A paid more than four times bidder C's payment, we take this as evidence that at least one additional bid was winning. In particular, we speculate that bidder C won with an additional bid. Winning a small package at low cost is consistent with the constant strategy and with CMRA-truthful bidding.

While we think it is likely that bidder B won with a headline demand, it is not clear

Lot	Description	Supply	R	Bidder A	Bidder B	Bidder C
CMRA						
B	2x5 MHz in the 700 MHz band	6	95	2	3	1
D	5 MHz in the 700 MHz band	4	25		4	
E	40 MHz in the 2.3 GHz band	1	0			
F	10 MHz in the 2.3 GHz band	6	25		6	
Non-competitive						
A	2x10 MHz in the 900 MHz band	3	0	1	1	1
Expenditure						
	In million DKK		820	485	1620	107.6

Table OA.1: Supply and auction outcome in the 2019 Danish spectrum auction
Notes: R = reserve price in million DKK; the E lot came with a coverage obligation

whether bidder A won with a headline or an additional bid. Bidder B paid DKK 1135 million more than bidder A for also winning another B lot, the four D lots, and the six F lots. If bidder B won the headline demand, then $1135 = p_B + 4p_D + 6p_F$. Suppose bidder A won the headline demand, implying $p_B = 485.2/2 = 242.6$ and $892.4 = 4p_D + 6p_F$. Moreover, assume that $p_D = p_F$ as the D and F lots have the same reserve price. Linear prices then imply that $p_D = p_F = 89.24$, which is not implausible. Conversely, if bidder B won their headline demand and final prices for D and F were, say, about 65, then this would imply $p_B = 485$. In particular, bidder A would have bought two B lots with an additional bid at half price. Hence, we cannot rule out the possibilities that bidder A won their headline demand or with an additional bid. Finally, we do not see any signs that the auctioned ended as in the risk-free demand reduction equilibrium.

OA.2   Details on the 2021 Danish auction

The process was similar to the two previous auctions. Bidders first had the chance to obtain 2x10 MHz in the 2.1 GHz band with a coverage obligation for a reserve price of 0 (2.1-D lot). All bidders bought such a license. Table OA.2 summarizes the supply and outcome.

There were two subsequent CMRAs. In the first CMRA, there were ten lots in the 1500 MHz band available: a single 25 MHz (lot 1.5-B) for a reserve price of DKK 10 million, eight 5 MHz lot (1.5-M) for a reserve price of DKK 10 million each, and another single 25 MHz block (1.5-T) for a reserve price of DKK 10 million. In the 2.1 GHz spectrum, there were six 2x5 MHz blocks (2.1-U) available for a reserve price of DDK 25 million each. In the 2.3 GHz band, there were two lots for 20 MHz available for a reserve price of DDK 25 million. In the 3.5 GHz band there were three categories of lots. First, there were three lots in the 3.5 GHz band available (3.5-D). The reserve price for such a lot was DDK 75 million. One such lot corresponds to 80 MHz in the 3.5 GHz spectrum

and 400 MHz in the 26 GHz spectrum. Second, there was a single lot of 60 MHz (3.5-P) for a reserve price of DDK 25 million in the 3.5 GHz band with a leasing obligation. Third, there were nine 10 MHz lots (3.5-U) for a reserve price of DDK 25 million. The second CMRA was for the remaining lots in the 26 GHz band. In total there were 2850 MHz unpaired frequencies in the 26 GHz band. After subtracting the 1200 MHz sold in the first auction through the 3.5-D lots, there were 1650 MHz available (in lots of 200 MHz and 250 MHz) in the second CMRA. The reserve price was about DKK 5 million per lot.

Each of the three bidders won a 2x10 MHz in the 2.1 GHz band with a coverage obligation for a reserve price of 0, two 2x5 MHz lots in the 2.1 GHz band, and a 3.5-D lot (80 MHz in the 3.5 GHz band and 400 MHz in the 26 GHz band).

In addition, bidder A won 40 MHz in the 3.5 GHz band (four 3.5-U lots) in the first CMRA and 600 MHz in the 26 GHz band in the second CMRA. Bidder A's total payment was DKK 540,525,000.

In addition to the above, bidder B won the 1.5-B lot, four 1.5-M lots, the two lots in the 2.3 GHz band, and 50 MHz in the 3.5 GHz band (five 3.5-U lots). In the second CMRA, bidder B won 850 MHz in the 26 GHz band. Bidder B's total payment was DKK 794,685,000.

In addition to the above, bidder C won the 1.5-T lot, four 1.5-M lots, and the 60 MHz in the 3.5 GHz spectrum with the leasing obligation. In the second CMRA, bidder C won 200 MHz in the 26 GHz band. Bidder C's total payment was DKK 740,976,000.

We first look at the differences between bidders B and C. Bidder B won the two 2.3-U lots in the 2.3 GHz band (40 MHz in total), 10 MHz less in the 3.5 GHz band (but without the leasing obligation), and 650 MHz more in the 26 GHz band. Bidder B paid DKK 53,709,000 more than bidder C. Bidder B's final assignment seems to dominate bidder C's and cost only DKK 54 million more. Compare this number to the reserve price of DKK 100 million for the 2.3 GHz band alone. Hence, we suspect that bidder B used additional bids to win the large package (as under CMRA-truthful bidding with decreasing marginal values).

Next, we compare the outcomes of bidders A and B. Bidder B paid DKK 254 million more than bidder A and got the additional 45 MHz in the 1500 MHz band, 40 MHz in the 2300 MHz band, 10 MHz in the 3.5 GHz band, and 250 MHz in the 26 GHz band. The reserve price for the additional lots won by bidder B is DKK 180 million. Hence, bidder B paid DKK 74 million in excess of the reserve price.

Comparing bidders A and C, bidder C won 45 MHz in the 1500 MHz band while bidder A did not win any lot in this category. Bidder C won 20 MHz more in the 3.5 GHz band (but subject to the leasing obligation), and 400 MHz less in the 26 GHz band. Bidder A paid DKK 200 million less, however. The reserve price of the 45 MHz in the 1500 MHz band was DDK 50 million.

Lot	Description	Supply	R	Bidder A	Bidder B	Bidder C
CMRA						
1.5-B	25 MHz in the 1500 MHz band (bottom)	1	10		1	
1.5-M	5 MHz in the 1500 MHz band	8	10		4	4
1.5-T	25 MHz in the 1500 MHz band (top)	1	10			1
2.1-U	2x5 MHz in the 2.1 GHz band	6	25	2	2	2
2.3-U	20 MHz in the 2.3 GHz band	2	50		2	
3.5-D	80 MHz in 3.5 GHz + 400 MHz in 26 GHz	3	75	1	1	1
3.5-P	60 MHz in 3.5 GHz (leasing obligation)	1	25			1
3.5-U	10 MHz in the 3.5 GHz band	9	25	4	5	
26-U	200 MHz/250 MHz in the 26 GHz band	8	5	3	4	1
Non-competitive						
2.1-D	2x10 MHz in the 2.1 GHz band	3	0	1	1	1
Expenditure						
	In million DKK		865	541	795	741

Table OA.2: Supply and auction outcome in the 2021 Danish spectrum auction
Note: R = reserve price in million DKK

We conclude that it is likely that bidder B won with an additional bid. Due to the many prices, we cannot say whether bidders A and B won their headline demands or with additional bids. There is, however, no evidence for risk-free demand reduction.